Research Project Report

(k, t)-choosability of graphs

by

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ABSTRACT

A (k,t)-list assignment L of a graph G is a mapping which assigns a set of size k to each vertex v of G and $|\bigcup_{v\in V(G)} L(v)| = t$. A graph G is (k,t)-choosable if G has a proper coloring f such that $f(v) \in L(v)$ for each (k,t)-list assignment L.

In 2011, Charoenpanitseri, Punnim and Uiyyasathian gave a characterization of (k,t)-choosability of n-vertex graphs when $t \geq kn - k^2 - 2k + 1$ and left open problems when $t \leq kn - k^2 - 2k$ Recently, Ruksasakchai and Nakprasit obtain the results when $t = kn - k^2 - 2k$. In this research report, we extend the results to case $t = kn - k^2 - 2k - 1$.

Keywords : coloring, list assignment, choosable, choosability

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CHAPTER I

INTRODUCTION

1.1 Rationale and Background of the Study

Recall some known definitions and notations here. Unless we say otherwise, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. A *clique* is a set of pairwise adjacent vertices in a graph; a clique of size k is called a k-*clique*. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle; the cycle with n vertices is denoted by C_n. A *complete graph* is a graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called *partite sets*. A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets; the complete bipartite graph with partite sets of size a and b is denoted by $K_{a,b}$. Given a graph G and $S \subseteq V(G)$, $G - S$ is the graph obtained from G by deleting all vertices of S. In case $S = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph induced by* S, denoted by $G[S]$ is the graph obtained from G by deleting all vertices of $V(G)$ outside S. Given a graph H, a graph is said to be H-*free* if H is not its induced subgraph. A graph is said to be a *triangle-free* if it does not contain a 3-clique. A *complement* of a graph G, denoted by $\overline{G}$, is the graph with the vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *join* of graphs G and H, written $G \vee H$, is the graph obtained from G and H by adding the edges between all vertices of G and all vertices of H.

A *coloring* of a graph G is a mapping from $V(G)$ to a set of colors S such that adjacent vertices receive distinct colors. If $|S| = t$, then such coloring is called a *t-coloring*. A graph is *t-colorable* if it has a t-coloring. The *chromatic number* of G, denoted by $\chi(G)$ is the smallest positive integer t such that G is t-colorable. A *list assignment* of a graph G is a mapping which assigns a set of colors, called a *list* to each vertex $v \in V(G)$. A list assignment L of a graph G is said to be a *k-list assignment* if $|L(v)| = k$ for all $v \in V(G)$. A k-list assignment L of a graph G is said to be a *(k,t)-list assignment* if $|\bigcup_{v\in V(G)} L(v)| = t$. Given a list assignment L of a graph G, a coloring f of G is an *L-coloring* of G if $f(v)$ is chosen from $L(v)$ for each vertex $v \in V(G)$. A graph is *L-colorable* if it has an L-coloring. Particularly, if L is a (k,k)-list assignment of a graph G, then any L-coloring of G is a k-coloring of G. A graph G is *(k,t)-choosable* if G is L-colorable for every (k,t)-list assignment L of G. If a graph G is (k,t)-choosable for each positive integer t then G is called *k-choosable*, and the smallest positive integer k satisfying this property is called the *list chromatic number* of G denoted by $\chi_l(G)$.

Example 1.1. Let L be the 2-list assignment of C_5 as shown in Figure 1.1. That is, $L(v_1) = \{1,2\}$, $L(v_2) = \{1,3\}$, $L(v_3) = \{1,2\}$, $L(v_4) = \{2,3\}$ and $L(v_5) = \{1,3\}$. Because of $|\bigcup_{v\in V(C_5)} L(v)| = 3$, L is called a $(2,3)$-list assignment of C_5.

Let f be a coloring of C_5 as shown in Figure 1.1. That is, $f(v_1) = 2$, $f(v_2) = 3$, $f(v_3) = 1$, $f(v_4) = 3$ and $f(v_5) = 1$. Because of $f(v) \in L(v)$ for all $v \in V(C_5)$, f is an L-coloring of C_5.

If there is no ambiguous, each list is written without commas and braces;

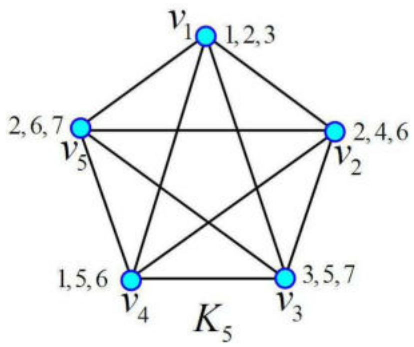

Figure 1.1: A $(2,3)$-list assignment of C_5.

moreover, each box containing a color from each list represent its coloring in order to simplify our figure. Figure 1.2 is the simplify figure of Figure 1.1.

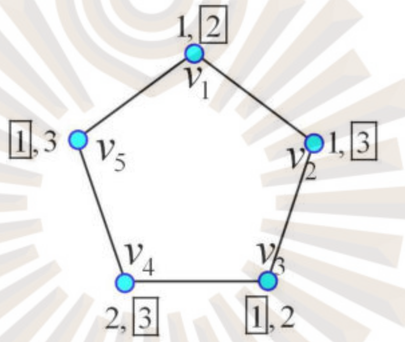

Figure 1.2: A $(2,3)$-list assignment of C_5.

Now, we consider $(2,3)$-choosability of C_5. The set of all $(2,3)$-list assignments of C_5 is divided into eight cases. $L_1, L_2, \ldots, L_8$ in Figure 1.3 represent a $(2,3)$-list assignments of C_5 in each case

The $(2,3)$-list assignment L_1 contains four vertices with the same list while L_2, L_3, L_4 and L_5 contain three vertices with the same list. The list assignments L_6, L_7 and L_8 contain only two vertices with the same list. It is shown in Figure 1.3 that C_5 is L_i-colorable for each $i = 1, 2, \ldots, 8$.

□

Example 1.2. Let G be the graph with eight vertices in Figure 1.4. The minimum

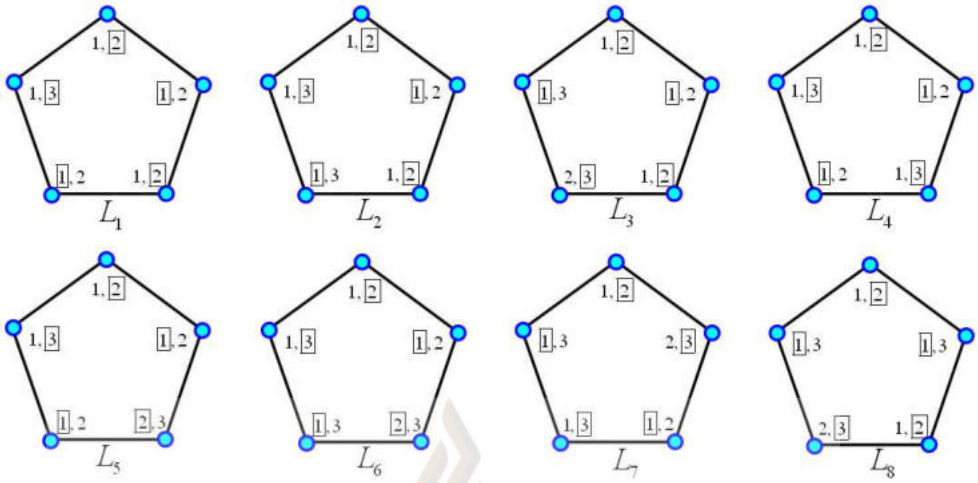

Figure 1.3: C_5 is $(2, 3)$-choosable.

number of colors in a 3-list assignment of G occurs when all vertices are assigned by the same list of size 3 while the maximum number of colors in a 3-list assignment of G occurs when all vertices are assigned by mutually disjoint lists as shown in Figure 1.4.

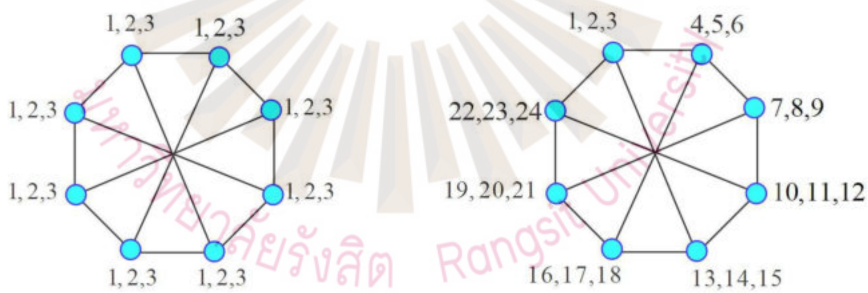

Figure 1.4: A $(3, 3)$-list assignment and a $(3, 24)$ list assignment

□

Unless we say otherwise, our parameters k, n and t in this dissertation are always positive integers such that $t \geq k$ and $t \leq kn$ because when $t < k$ or $t > kn$, there is no (k, t)-list assignment of a graph with n vertices, so it is automatically (k, t)-choosable. If $k \geq n$ then all graphs with n vertices are (k, t)-choosable. Besides, when $k \geq \chi_l(G)$, a graph G is always (k, t)-choosable; therefore, we focus on a positive integer k such that $k < \chi_l(G)$.

Let $S \subseteq V(G)$. If L is a list assignment of G, we let $L|_S$ denote L restricted to S and $L(S)$ denote $\bigcup_{v\in S} L(v)$. For a color set A, let $L-A$ be the new list assignment obtained from L by deleting all colors in A from $L(v)$ for each $v \in V(G)$. When A has only one color a, we write $L-a$ instead of $L-\{a\}$. Examples are illustrated in Figures 1.5.

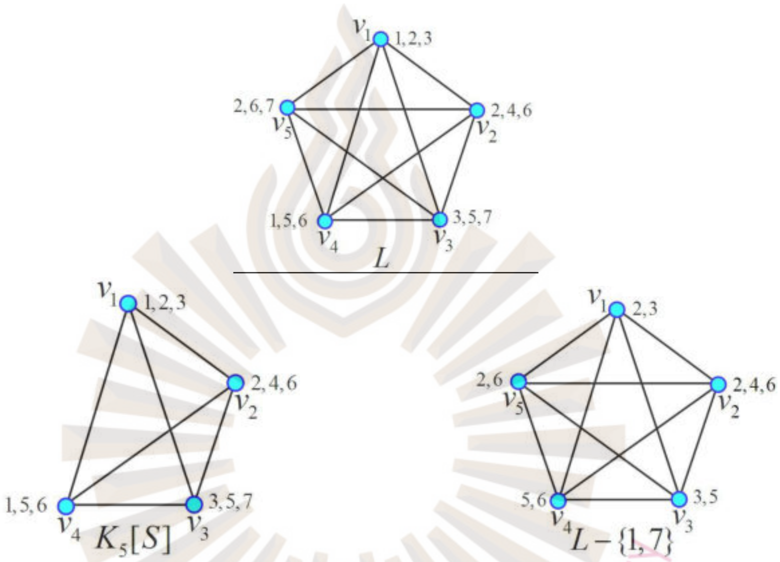

Figure 1.5: the list assignment $L|_S$ of $K_5[S]$ where $S = \{v_1, v_2, v_3, v_4\}$ and the list assignment $L-\{1,7\}$ of K_5

1.2 Objectives

1. Study knowledge in the field of graph theory, especially in graph coloring.

2. Find examples or applications of graph coloring.

3. Obtain new theorems in graph coloring.

1.3 Hypothesis of the research

If $t = kn - k^2 - 2k - 1$, then all n-vertex graph not containing K_{k+1} and $K_{k-2} \vee C_5$ is always (k,t)-choosable.

1.4 Scope of the research

Obtain enough results to present in international conference or public a paper in international journal in Scopus or ISI database.

1.5 Framework of the research

Study recent research related to (k,t)-choosability of graphs and try to obtain new theorems. Here, we focus on obtain a new theorem when $t = kn - k^2 - 2k - 1$ sectionBenefits expected to be received

1. Collect knowledge related to this field.
2. Obtain examples or applications of graph coloring in the real life
3. Obtain new theorems in graph coloring in order to present in international conference or public a paper in international journal in Scopus or ISI database.

CHAPTER II

LITERATURE REVIEW

2.1 Definitions and notations

Throughout the research subject, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of G. For $X \subseteq V(G)$, $G - X$ is the graph obtained from deleting all vertices of X from G. In case $X = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph induced by* X, denoted by $G[X]$ is the graph obtained from deleting all vertices of $V(G)$ outside X. The notaion $d(v)$ stands for the degree of v in G. For a subgraph H of G, $d_H(v)$ stands for the degree of v in H.

Let $S \subseteq V(G)$. If L is a list assignment of G, we let $L|_S$ denote L restricted to S and $L(S)$ denote $\bigcup_{v \in S} L(v)$. For a color set A, let $L - A$ be the new list assignment obtained from L by deleting all colors in A from $L(v)$ for each $v \in V(G)$. When A has only one color a, we write $L - a$ instead of $L - \{a\}$.

Example 2.1. *The cycle C_n is $(2, t)$-choosable unless n is odd and $t = 2$.*

Note that a graph G is $(2, 2)$-choosable if and only if G is 2-colorable. Hence, C_n is $(2, 2)$-choosable if and only if n is even. It remains to show that all of the cycles are $(2, t)$-choosable for $t \geq 3$.

Let $t \geq 3$ and L be a $(2, t)$-list assignment of C_n. Thus there are two adjacent vertices $v_1, v_n \in V(G)$ such that $L(v_1) \neq L(v_n)$. Let $v_2, v_3 \ldots, v_{n-1}$ be remaining vertices along the cycle C_n where v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. First we assign v_1 a color c in $L(v_1)$ which is not in $L(v_n)$ and then we assign vertex v_2

a color in $L(v_2)$ different from c and so on. This algorithm guarantees that each pair of adjacent vertices receives distinct colors.

2.2 History

A graph G is called *k-colorable* if every vertex of G can be labeled by at most k colors and every adjacent vertices receives distinct colors. The smallest number t such that G is t-colorable is called the *chromatic number* of G, denoted by $\chi(G)$. A *k-list assignment* L of a graph G is a function which assigns a set of size k to each vertex v of G. A *(k,t)-list assignment* of a graph is a k-list assignment with $|\bigcup_{v\in V(G)} L(v)| = t$. Given a list assignment L, a proper coloring f of G is an *L-coloring* of G if $f(v)$ is chosen from $L(v)$ for each vertex v of G. A graph G is *L-colorable* if G has an L-coloring. Particularly, if L is a (k,k)-list assignment of G, then any L-coloring of G is a *k-coloring* of G. A graph G is *(k,t)-choosable* if G is L-colorable for every (k,t)-list assignment L. If a graph G is (k,t)-choosable for each positive number t then G is called *k-choosable* and the smallest number k satisfying this property is called the *list chromatic number* of G denoted by $ch(G)$.

List coloring is a well-known problem in the field of graph theory. It was first studied by (Vizing, 1976) and by (Erdős, Rubin, and Taylor, 1979). They give a characterization of 2-choosable graphs. For $k \geq 3$, there is no characterization of k-choosable graphs. There are only results for some classes of graphs. For example, all planar graphs are 5-choosable, while some planar graphs are 3-choosable. See the following list for more information; (Thomassen, 1994) (Thomassen, 1995) (Zhang, and Xu, 2004) (Zhang, 2005) (Lam, Shiu, and Song, 2005) (Zhang, Xu, and Sun 2006) and (Zhu, Lianying, and Wang 2007).

In order to simplify the problem, (k,t)-choosability is defined. It is a partial problem of k-choosability. Instead of proving a graph can always be colored for entire k-list assignments, we prove the graph can be colored for k-list assignments that have exactly t colors. In 2011, (k,t)-choosability of graphs was explored by (Charoenpitseri, Punnim, and Uiyyasathian, 2011). They proved the following theorem.

Theorem 2.2. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) For an n-vertex graph G, if $t \geq kn - k^2 + 1$ then G is (k,t)-choosable.*

Moreover, they showed that the bound is best possible by proving if $t \leq kn - k^2$, then an n-vertex graph containing K_{k+1} is not (k,t)-choosable. Furthermore, they keep investigating the (k,t)-choosability to obtain another interesting theorem.

Theorem 2.3. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) Let $k \geq 3$. A K_{k+1}-free graph with n vertices is (k,t)-choosable for $t \geq kn - k^2 - 2k + 1$.*

Again, they showed that the bound is best possible for K_{k+1}-free graphs with n vertices by proving an n-vertex graph containing $C_5 \vee K_{k-2}$ is not (k,t)-choosable for $t \leq kn - k^2 - 2k$. In conclusion, they gave a characterization of (k,t)-choosability of n-vertex graphs when $t \geq kn - k^2 - 2k + 1$.

In 2013, Ruksasakchai and Nakprasit gave a characterization of $(k, kn - k^2 - 2k)$-choosability of n-vertex graphs as shown in the following theorem.

Theorem 2.4. *(Ruksasakchai, and Nakprasit, 2013) Let G be a graph with n vertices and $k \geq 3$. If G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k$*

Results on $(2,t)$-choosability of n-vertex graphs are almost completed by (Ruksasakchai, and Nakprasit, 2013) and (Charoenpanitseri, 2013). Here, we focus on $k \geq 3$. We will prove that if an n-vertex graph G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k - 1$.

Theorem 2.5 and Lemma 2.6 will be combined to obtain a result on (k,t)-choosability of graphs.

Theorem 2.5. *(He et al., 2008) Let L be a list assignment of a graph G and let $S \subseteq V(G)$ be a maximal non-empty subset such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$-colorable then G is L-colorable.*

Lemma 2.6. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) Let $A_1, A_2, \ldots, A_n$ be k-sets and $J \subseteq \{1, 2, \ldots, n\}$. If $|\bigcup_{i=1}^{n} A_i| \geq p$, then $|\bigcup_{i \in J} A_i| \geq p - (n - |J|)k$.*

The following statements appear in (Charoenpitseri, Punnim, and Uiyyasathian, 2011) and (Ruksasakchai, and Nakprasit, 2013). The authors apply them to obtain characterizations of (k,t)-choosability of n-vertex graphs. We also need the tools, as well.

Lemma 2.7. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) Let G be an n-vertex graph. If $k \geq n-2$ and G is K_{k+1}-free, then G is (k,t)-choosable for any positive integer k.*

Lemma 2.8. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) If a $(k+3)$-vertex graph is K_{k+1}-free, then it is (k,t)-choosable for $t \geq k+1$.*

Lemma 2.9. *(Ruksasakchai, and Nakprasit, 2013) Let G be an n-vertex graph where $n \geq 6$. If G does not contain K_{n-2} and $C_5 \vee K_{n-5}$, then G is $(n-3)$-colorable.*

Lemma 2.10. *(Ruksasakchai, and Nakprasit, 2013) Let G be a graph in Figure 2.1. If each list has size 2 except exactly two lists of size 3 and $|L(x_4)| = 2$ or $|L(x_5)| = 2$, then G is L-colorable.*

Lemma 2.11. *(Ruksasakchai, and Nakprasit, 2013) Let G be a graph with 7 vertices and $\chi(G) = 4$. If G does not contain K_4 and $C_5 \vee K_1$, then $\delta(G) \geq 3$ and $\Delta(G) = 4$.*

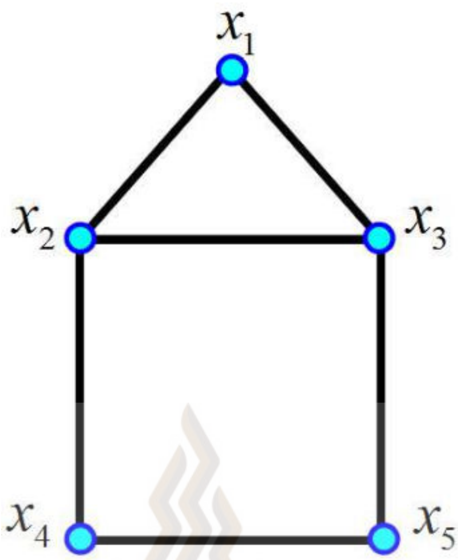

Figure 2.1: The graph for Lemma 2.7

Corollary 2.12. *(Ruksasakchai, and Nakprasit, 2013) If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4, then G is $(3,6)$-choosable.*

Recently, Ohba's conjecture is proved by Noel as shown in Theorem 2.13.

Theorem 2.13. *(Noel, 2013) If $|V(G)| \leq 2\chi(G)+1$, then $ch(G) = \chi(G)$.*

The theorem is powerful because several interesting results can be obtained; for example, Corollary 2.14 and Corollary 2.15.

Corollary 2.14. *Let G be an n-vertex graph with $n \leq 7$. If G is 3-colorable, then G is 3-choosable.*

Proof. Let G be an n-vertex graph with $n \leq 7$. Assume that G is 3-colorable. Then $\chi(G) \leq 3$. Then we add some edges to G to obtain a graph H such that $\chi(H) = 3$. Since $|V(H)| \leq 7 = 2\chi(H)+1$, we obtain that $ch(H) = \chi(H) = 3$ by Theorem 2.13. Hence G is 3-choosable because G is a subgraph of H □

Corollary 2.15. *Let G be an n-vertex graph with $n \leq 9$. If G is 4-colorable, then G is 4-choosable.*

Proof. Let G be an n-vertex graph with $n \leq 9$. Assume that G is 4-colorable. Then $\chi(G) \leq 4$. Then we add some edges to G to obtain a graph H such that $\chi(H) = 4$. Since $|V(H)| \leq 9 = 2\chi(H) + 1$, we obtain that $ch(H) = \chi(H) = 4$ by Theorem 2.13. Hence G is 4-choosable because G is a subgraph of H □

Finally, we need one more theorem and one more lemma to prove our main results.

Theorem 2.16. *(Brook, 1941) If G is a graph other than odd cycle and complete graph, then* $\chi(G) \leq \Delta(G)$.

Lemma 2.17. *(Ronald, and Robin, 1999) (Ruksasakchai, and Nakprasit, 2013) Let G be a graph with* $\delta(G) \geq 3$ *and* $\Delta(G) = 4$. *If G have no* K_4 *and* $C_5 \vee K_1$, *then G must be one of the 7 graphs in Figure 2.2.*

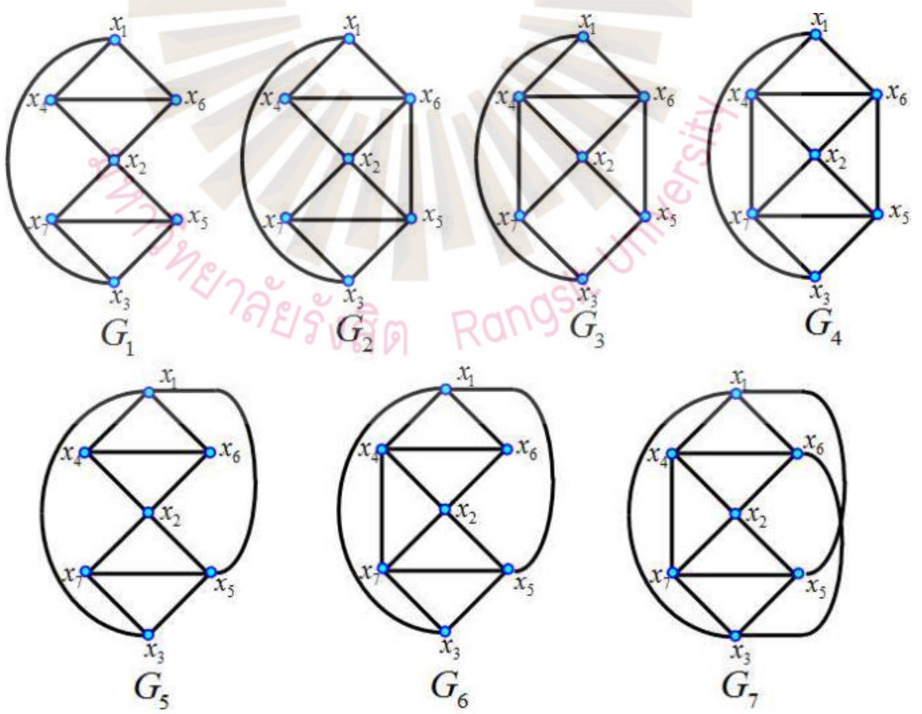

Figure 2.2: 7-vertex graph with chromatic number 4 and having no K_4 and $C_5 \vee K_1$

2.3 Basic properties and examples

When we try to color all vertices of a graph with some conditions, it tends to success and be easier if we have more colors. However, this is not true for a (k,t)-list assignment. It may not be true that (k,t)-choosability implies $(k,t+1)$-choosability. Example 2.18 illustrates this fact.

Example 2.18. Let X, Y be the bipartite sets of $K_{10,10}$. To show that $K_{10,10}$ is $(3,4)$-choosable, let L be a $(3,4)$-list assignment of $K_{10,10}$. For any $u \in X$, at least one of the numbers $1, 2$ is in $L(u)$. Hence, each vertex in X can be colored by only color 1 or 2. For all $v \in Y$, at least one of the numbers $3, 4$ is in $L(v)$. Hence, we can color each vertex in Y by only color 3 or 4.

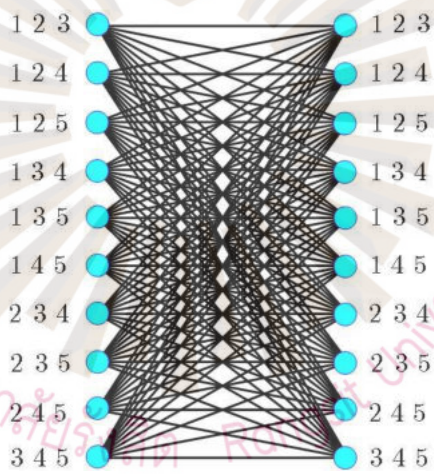

Figure 2.3: A $(3,5)$-list assignment of $K_{10,10}$

To show that $K_{10,10}$ is not $(3,5)$-choosable, let L be the $(3,5)$-list assignment as shown in Figure 2.3. At least three colors must be used to color all vertices in each partite set of $K_{10,10}$. However, only five colors are available; hence, there are $u \in X$ and $v \in Y$ receiving the same color. It is a contradiction.

□

Although (k,t)-choosability does not imply $(k,t+1)$-choosability, if the number t is large enough, we can prove that (k,t)-choosability implies $(k,t+1)$-choosability. Theorem 2.19 gives the number of colors we need to guarantee this statement.

Theorem 2.19. *Let G be an n-vertex graph. If G is L_1-colorable for every k-list assignment L_1 such that $|\bigcup_{v\in V(G)} L_1(v)| = t$ and $n\binom{k}{2} < \binom{t+1}{2}$, then G is L_2-colorable for every k-list assignment L_2 such that $|\bigcup_{v\in V(G)} L_2(v)| \geq t$.*

Proof. Since $n\binom{k}{2} < \binom{t+1}{2}$, there exists a pair of colors which does not appear together in a list, say $1, 2$. Then we construct a k-list assignment L_1 defined by

$$L_1(v) = \begin{cases} L_2(v) & \text{if } 1 \in L_2(v), \\ L_2(v) \cup \{1\} - \{2\} & \text{if } 2 \in L_2(v). \end{cases}$$

Since G is not L_2-colorable, G is not L_1-colorable. □

Definition 2.20. *Given a collection of sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, a System of Distinct Representatives (SDR) of $\mathcal{A}$ is a set of distinct elements $a_1, a_2, \ldots a_n$ such that $a_i \in A_i$ for all i.*

The following theorem shows the well-known necessary and sufficient condition for the existence of an SDR. Indeed, Hall's Theorem is originally proved in the language of an SDR and is equivalent to Manger's Theorem .

Theorem 2.21. *Given a collection of sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, an SDR of $\mathcal{A}$ exists if and only if $|\bigcup_{i\in J} A_i| \geq |J|$ for all $J \subset \{1, 2, \ldots, n\}$.*

Corollary 2.22. *Let L be a list assignment of a graph G. If $|L(S)| \geq |S|$ for all $S \subset V(G)$, then G is L-colorable. Moreover, there exists an L-coloring such that each vertex of G assigned by distinct colors.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. From Theorem 2.21, there exist $c_1 \in L(v_1), c_2 \in L(v_2), \ldots, c_n \in L(v_n)$ such that $c_1, c_2, \ldots, c_n$ are distinct. Thus we define $f : V(G) \to \{1, 2, \ldots, n\}$ by $f(v_i) = c_i$; hence, f is an L-coloring. □

CHAPTER III

RESEARCH METHODOLOGY

3.1 Study

First, we will read several books in the field of graph theory, especially in graph coloring in order to obtain their techniques, perspectives and how they think in order to find a way to do research in this field. Second, we will search knowledge related to this field to conclude them and combine to knowledge obtained from books.

The following books are the books that we will study.

- The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of its Creators Soifer, Alexander.
- Problem-Solving Methods in Combinatorics: An Approach to Olympiad Problems Sobern, Pablo
- Chromatic Graph Theory (Discrete Mathematics and Its Applications) Chartrand, Gary
- An Atlas of Graphs (Mathematics) Read, Ronald C.
- Erdos on Graphs: His Legacy of Unsolved Problems Chung, Fan
- Graph Theory As I Have Known It (Oxford Lecture Series in Mathematics and It's Applications) Tutte, W. T.
- Chromatic Polynomials and Chromaticity of Graphs F. M. Dong

- Graphs, Colourings and the Four-Colour Theorem (Oxford Science Publications) Wilson, Robert A.

3.2 Find and Prove

After we study books and recent paper, we will find a topic research. First, we combine all ideas to a conjecture and we try to prove them. If it is true, we finish them; otherwise, we go back to real books and papers in order to find a new conjecture. Repeat the process until obtain enough results.

3.3 Publication

After we obtain main theorems, we will present them in international conference or public a paper in international journal in Scopus or ISI database. The following are examples of mathematics journals in SCOPUS index.

- Acta Applicandae Mathematicae
- Acta Mathematica
- Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis
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- International Journal of Computational and Mathematical Sciences
- International Journal of Mathematics and Mathematical Sciences
- Journal of Discrete Mathematical Sciences and Cryptography
- Journal of Mathematical Sciences
- Malaysian Journal of Mathematical Sciences
- Missouri Journal of Mathematical Sciences

CHAPTER IV

MAIN RESULTS

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4.1 On $(2,t)$-choosability of graphs

We start with $(2,t)$-choosability of $K_{3,3}-e$, a domino and cycles.

Example 4.1. A bipartite graph $K_{3,3}-e$ is not $(2,t)$-choosable for $t=3,4,5$.

Proof. Suppose $t=3,4$ or 5. Let L be a $(2,t)$-list assignment of $K_{3,3}-e$ as shown in the Figure 4.1. if $a=2, b=2$ then $t=3$, if $a=2, b=4$ then $t=4$ and if $a=4, b=5$ then $t=5$.

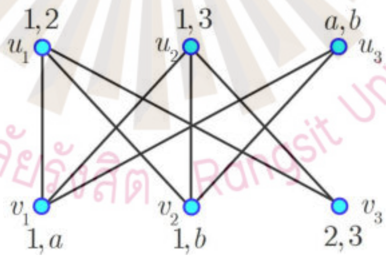

Figure 4.1: A $(2,t)$-list assignment L of $K_{3,3}-e$ where $t=3,4,5$

If u_1 and u_2 are labeled by color 1, the vertex v_3 cannot labeled. If u_1 or u_2 is labeled by color 1, then v_1 and v_2 must be labeled by color a and b, respectively. Consequently, the vertex u_3 cannot be labeled. Hence, $K_{3,3}-e$ is not L-colorable. Therefore, $K_{3,3}-e$ is not $(2,t)$-choosable for $t=3,4,5$.

□

Example 4.2. A domino is not $(2, t)$-choosable for $t = 3, 4$.

Proof. Suppose $t = 3$ or 4. Let L be a $(2, t)$-list assignment of a domino with the vertex set $v_1, v_2, \ldots, v_6$ as shown in the figure.

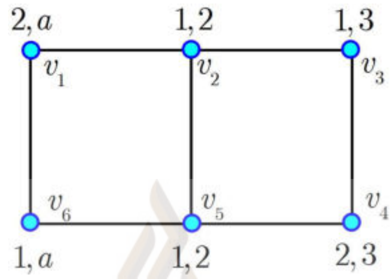

Figure 4.2: A 2-list assignment L of a domino where a is color 3 or color 4

If v_2 is labeled by color 1, then v_3 and v_5 must be labeled by color 3 and color 2, respectively. Hence v_4 cannot be labeled. If v_2 is labeled by color 2, then v_1 and v_5 must be labeled by color a and color 1, respectively. Hence v_6 cannot be labeled. That is, the domino is not L-colorable. Therefore, G is not $(2, t)$-choosable for $t = 3, 4$. □

(Charoenpitseri, Punnim, and Uiyyasathian, 2011) give a complete result on (k, t)-choosablity of an n-vertex graph containing K_{k+1}. Particulary, a complete result on $(2, t)$-choosability of an n-vertex graph containing a triangle is revealed as shown in Theorem 4.3.

Theorem 4.3. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) Let G be an n-vertex graph. If G contains a triangle, then it is not $(2, t)$-choosable for $t \leq 2n-4$. If G does not contain a triangle, then it is $(2, t)$-choosable for $t \geq 2n-6$.*

Before going to our main results, we will introduce some tools using in our proof. Theorem 4.4 and Lemma 4.5 are applied when we prove that a graph is $(2, t)$-choosable for some number t while Lemma 4.6 is applied when we prove that a graph is not $(2, t)$-choosable for some number t.

Let $S \subseteq V(G)$. If L is a list assignment of G, we let $L|_S$ denote L restricted to S and $L(S)$ denote $\bigcup_{v \in S} L(v)$.

Theorem 4.4. *Let L be a list assignment of a graph G and let $S \subseteq V(G)$ be such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$-colorable then G is L-colorable.*

Lemma 4.5. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) Let $A_1, A_2, \ldots, A_n$ be k-sets and $J \subseteq \{1, 2, \ldots, n\}$. If $|\bigcup_{i=1}^{n} A_i| \geq p$, then $|\bigcup_{i \in J} A_i| \geq p - (n - |J|)k$.*

Lemma 4.6. *Let H be an m-vertex subgraph of an n-vertex graph G. If H is not $(2, t_0)$-choosable, then G is not $(2, t)$-choosable for $t_0 \leq t \leq 2n - 2m + t_0$.*

Proof. Let H be an m-vertex subgraph of an n-vertex graph G. Let t_0, t be numbers such that $t_0 \leq t \leq 2n - 2m + t_0$. Assume that H is not $(2, t_0)$-choosable. Hence, there is a $(2, t_0)$-list assignment L_0 such that H is not L_0-colorable. Then we extend a $(2, t_0)$-list assignment L_0 of H to a $(2, t)$-list assignment L of G by assigning the remaining colors to the remaining vertices outside $V(H)$. Notice that G has $n - m$ remaining vertices and L has $t - t_0$ remaining colors. The condition $t - t_0 \leq 2n - 2m$ can confirm the existence of L. Since H is not L_0-colorable, G is not L-colorable. Consequently, G is not $(2, t)$-choosable. □

4.2 Main results

In (Wongsakorn), the authors show that an n-vertex graph not containing a triangle is $(2, t)$-choosable for $t \geq 2n - 6$. Then we study $(2, t)$-choosability of a triangle-free graph when $t \leq 2n - 7$. The first result is that an n-vertex graph containing $K_{3,3} - e$ is not $(2, t)$-choosable for $3 \leq t \leq 2n - 7$.

Theorem 4.7. *An n-vertex graph containing $K_{3,3} - e$ is not $(2, t)$-choosable for $3 \leq t \leq 2n - 7$.*

Proof. Let G be an n-vertex graph and $t \leq 2n-7$. By Example 4.1, $K_{3,3}-e$ is not $(2,t)$-choosable for $t = 3,4,5$. Consequently, G is not $(2,t)$-choosable for $t = 3,4,5$. Notice that $K_{3,3}-e$ is a 6-vertex subgraph of G and is not $(2,5)$-choosable. By Lemma 4.6, G is not $(2,t)$-choosable for $5 \leq t \leq 2(n-6)+5 = 2n-7$. □

Next, we focus on an n-vertex graph containing neither a triangle nor $K_{3,3}-e$. Let us introduce a theorem on $(2,t)$-choosability of a triangle-free graph.

Theorem 4.8. *(Charoenpitseri, Punnim, and Uiyyasathian, 2011) A triangle-free graph with n vertices is* $(2,2n-7)$*-choosable if and only if it does not contain* $K_{3,3}-e$ *as a subgraph.*

We apply the above theorem to obtain the second result in Theorem 4.9.

Theorem 4.9. *An n-vertex graph containing neither a triangle nor* $K_{3,3}-e$ *is* $(2,t)$*-choosable for* $t \geq 2n-7$.

Proof. Let G be an n-vertex graph containing neither a triangle nor $K_{3,3}-e$. If $t \geq 2n-6$, then G is $(2,t)$-choosable by Theorem 4.3. If $t = 2n-7$, then G is $(2,t)$-choosable by Theorem 4.8. □

Now, the result in case $t \geq 2n-7$ is revealed. Then we keep studying in the remaining case; the case that $t \leq 2n-8$. The third result is that every n-vertex graph containing a domino or C_5 is not $(2,t)$-choosable for $3 \leq t \leq 2n-8$.

Theorem 4.10. *An n-vertex graph containing a domino or* C_5 *is not* $(2,t)$*-choosable for* $3 \leq t \leq 2n-8$.

Proof. Let G be an n-vertex graph.

Case 1. G contains a domino as a subgraph. By Example 4.2, a domino is not $(2,t)$-choosable for $t = 3,4$. Clearly, G is not $(2,t)$-choosable for $t = 3,4$. Notice that a domino is a 6-vertex subgraph of G and it is not $(2,4)$-choosable. By

Lemma 4.6, G is not $(2,t)$-choosable for $t \leq 2(n-6)+4 = 2n-8$.

Case 2. G contains C_5 as a subgraph. Since C_5 is not bipartite, it is not $(2,2)$-choosable. Obviously, G is not $(2,2)$-choosable. Notice that C_5 is a 5-vertex subgraph of G and it is not $(2,2)$-choosable. By Lemma 4.6, G is not $(2,t)$-choosable for $t \leq 2(n-5)+2 = 2n-8$ □

Last, we study an n-vertex graph containing neither a triangle, $K_{3,3}-e$, a domino, C_5. The last result is that the graph is $(2,t)$-choosable for $t \geq 2n-8$.

Theorem 4.11. *If an n-vertex graph contains neither a triangle, $K_{3,3}-e$, a domino nor C_5, then it is $(2,t)$-choosable for $t \geq 2n-8$.*

Proof. Assume that an n-vertex graph G contains neither a triangle, $K_{3,3}-e$, a domino nor C_5 and $t \geq 2n-8$. Let L be a $(2,t)$-list assignment of G and let $S \subseteq V(G)$ be such that $|L(S)| < |S|$.

Recall that $|L(V(G))| = t \geq 2n-8$. By Lemma 4.5, $|L(S)| \geq (2n-8)-2(n-|S|) = 2|S|-8$. Then $|S| > |L(S)| \geq 2|S|-8$. Hence, $|S| \leq 7$.

Next, we will prove that $G[S]$ is $L|_S$-colorable in order to apply Theorem 4.4.

If $G[S]$ has a vertex of degree 1, we may successively delete vertices of degree 1 and consider only the remaining graph. Hence, we may suppose that $G[S]$ has no vertex of degree 1.

Case 1. $|S| \leq 6$. Since $G[S]$ contains neither a triangle nor C_5, $G[S]$ is bipartite. If $G[S]$ has 2 cycles, then $G[S]$ contains a domino. Hence, we conclude that $G[S]$ has only 1 cycle. Therefore, $G[S]$ is $(2,t)$-choosable for all t.

Case 2. $|S| = 7$. Since $t \geq 2n-8$, we obtain $|L(S)| \geq 6$. If $|L(S)| \geq 7$, then G is suddenly L-colorable by Theorem 4.4. Suppose that $|L(S)| = 6$. In other words, there are six colors. If a color c is available in exactly one vertex, then we label the vertex by using color c. According to the Case 1, the remaining six vertices can be labeled; hence, $G[S]$ is $L|_S$-colorable.

Suppose that all six colors available in at least two vertices. Recall that $G[S]$ has seven vertices and have six available colors. At least four colors are available in exactly two vertices. Since $G[S]$ does not contain a triangle, $G[S]$ has at most one vertex with degree at least four. Hence, one color from the four colors is available in two non-adjacent vertices, then we label the two vertices by using the color. Again, Case 1 confirms that the remaining five vertices can be labeled; hence, $G[S]$ is $L|_S$-colorable by Theorem 4.4. □

4.3 Applications

In this section, we apply our main results to some classes of graphs such as grid graphs and hypercube graphs. We start this section with definitions and examples of the two classes of graphs.

A *grid graph* is a unit distance graph corresponding to the square lattice, so that it is isomorphic to the graph having a vertex corresponding to every pair of integers (a, b), and an edge connecting (a, b) to $(a+1, b)$ and $(a, b+1)$. The finite grid graph $G(m, n)$ is an $m \times n$ rectangular graph isomorphic to the one obtained by restricting the ordered pairs to the range $0 \leq a < m, 0 \leq b < n$. A *domino* is $G(2, 3)$. (See examples in Figure 4.3.)

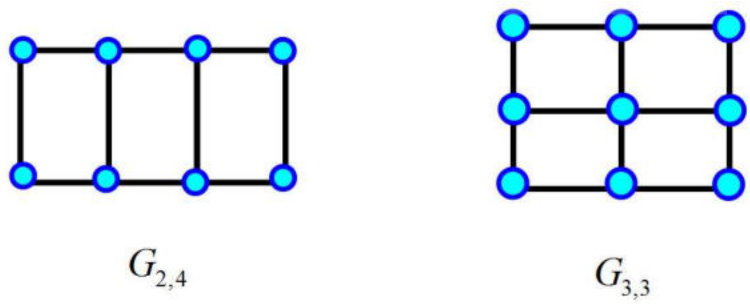

Figure 4.3: Examples of grid graphs

An *a-hypercube* graph, denoted by Q_a, is the graph whose vertices are the a

tuples with entries in $\{0,1\}$ and whose edges are the pair of a-tuples that differ in exactly one position. (See examples in Figure 4.4.)

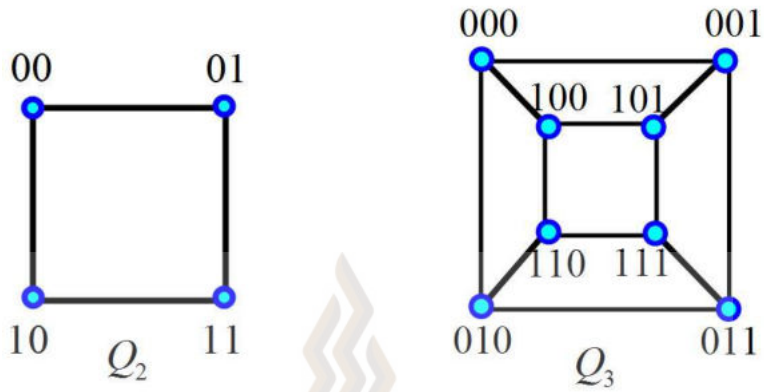

Figure 4.4: Examples of hypercubes

According to the four main result, $(2,t)$-choosability of some classes of graphs are obtained.

Remark 4.12. Let $a \geq 2$ and $b \geq 3$. A grid graph $G(a,b)$ is $(2,t)$-choosable if and only if $t = 2$ or $t \geq 2ab - 7$.

Proof. Let $a \geq 2, b \geq 3$ and $G(a,b)$ is a grid graph with ab vertices.

Case 1. $t = 2$. Then $G_(a,b)$ is $(2,t)$-choosable because $G_(a,b)$ is bipartite which is 2-colorable.

Case 2. $3 \leq t \leq 2ab - 8$. Because of $a \geq 2$ and $b \geq 3$, $G(a,b)$ contains a domino as a subgraph. Then $G(a,b)$ is not $(2,t)$-choosable by Theorem4.10.

Case 3. $t = 2ab - 7$. Notice that $G(a,b)$ contains neither a triangle nor $K_{3,3} - e$. Then G is $(2,t)$-choosable by Theorem 4.9.

Case 4. $t \geq 2ab - 6$. Notice that $G(a,b)$ does not contain a triangle as a subgraph. Then $G(a,b)$ is $(2,t)$-choosable by Theorem 4.3. □

Remark 4.13. An n-hypercube graph Q_a where $a \geq 3$ is $(2,t)$-choosable if and only if $t = 2$ or $t \geq 2^{a+1} - 7$

Proof. Let Q_a be an n-hypercube graph where $a \geq 3$. Then Q_a has 2^a vertices.

Case 1. $t = 2$. Then Q_a is $(2,t)$-choosable because Q_a is bipartite which is 2-colorable.

Case 2. $3 \leq t \leq 2ab - 8$. Observe that Q_a contains a domino as a subgraph. Then Q_a is not $(2,t)$-choosable by Theorem 4.10.

Case 3. $t = 2ab - 7$. Observe that Q_a contains neither a triangle nor $K_{3,3} - e$. Then Q_a is $(2,t)$-choosable by Theorem 4.9.

Case 4. $t \geq 2ab - 6$. Recall that G does not contain a triangle as a subgraph. Then Q_a is $(2,t)$-choosable by Theorem 4.3. □

A complete result of $(2,t)$-choosability is obtained not only for grid graphs and hypercube graphs but also for all classes of graphs containing a domino.

Theorem 4.14. *Let G be an n-vertex triangle-free and $K_{3,3} - e$-free graph containing a domino and $t \geq 3$. Then G is $(2,t)$-choosable if and only if $t \geq 2n - 7$.*

Proof. Let G be an n-vertex triangle-free and $K_{3,3} - e$-free graph containing a domino and $t \geq 3$.

Case 1. $t \leq 2n - 8$. Recall that G contains a domino as a subgraph. Then G is not $(2,t)$-choosable by Theorem 4.10.

Case 2. $t = 2n - 7$. Recall that G is an n-vertex triangle-free and $K_{3,3} - e$-free graph. Then G is $(2,t)$-choosable by Theorem 4.9.

Case 3. $t \geq 2n - 6$. Recall that G does not contain a triangle as a subgraph. Then G is $(2,t)$-choosable by Theorem 4.3. □

Similarly, a complete result of classes of graphs containing C_5 is obtained.

Theorem 4.15. *Let G be an n-vertex triangle-free and $K_{3,3} - e$-free graph containing C_5 and $t \geq 3$. Then G is $(2,t)$-choosable if and only if $t \geq 2n - 7$.*

Proof. Let G be an n-vertex triangle-free and $K_{3,3}-e$-free graph containing C_5 and $t \geq 3$.

Case 1. $t \leq 2n-8$. Recall that G contains C_5 as a subgraph. Then G is not $(2,t)$-choosable by Theorem 4.10.

Case 2. $t = 2n-7$. Recall that G is an an n-vertex triangle-free and $K_{3,3}-e$-free graph. Then G is $(2,t)$-choosable by Theorem 4.9.

Case 3. $t \geq 2n-6$. Recall that G does not contain a triangle as a subgraph. Then G is $(2,t)$-choosable by Theorem 4.3. □

4.4 On (k,t)-choosability of graphs

The main result is in Theorem 4.22. In order to prove the main result, we need to prove that Theorem 4.17 and Theorem 4.21. Lemma 4.16 is established for Theorem 4.17 while Lemma 4.18, Lemma 4.19 and Lemma 4.20 are established for Theorem 4.21.

Lemma 4.16. *Let G be an 8-vertex graph with $\Delta(G) = 5$. If G has no $C_5 \vee K_2$ and K_5, then G is 4-colorable.*

Proof. Let G be an 8-vertex graph with $\Delta(G) = 5$. Assume that G has no $C_5 \vee K_2$ and K_5.

If there is a vertex v such that $d(v) \leq 3$, then $G - v$ is 4-colorable by Lemma 2.9; hence, G is 4-colorable, as well. Suppose that $\delta(G) \geq 4$. Let v_1 be a vertex with $d(v_1) = 5$ of G and v_2 be a vertex which is not adjacent to v_1. Notice that $d(v_2)$ is 4 or 5.

Case 1. $G - v_1 - v_2$ contains $C_5 \vee K_1$. Under the condition $\Delta(G) = 5$, $G - v_1 - v_2$ must be $C_5 \vee K_1$. Then G must be a subgraph of the graph shown in Figure 4.5. According to the figure, G and its subgraphs are 4-colorable.

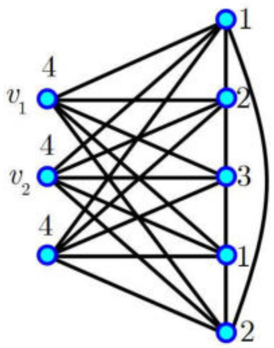

Figure 4.5: Case 1

Case 2. $G - v_1 - v_2$ contains K_4. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$, $H = G - \{u_1, u_2, u_3, u_4\}$ and $V(H) = \{v_1, v_2, w_1, w_2\}$.

Case 2.1. $d(v_2) = 5$. Notice that H has at most 5 edges because H has 4 vertices and v_1, v_2 are not adjacent. Then $d_H(v_1) + d_H(v_2) + d_H(w_1) + d_H(w_2) \leq 10$. However, $d(v_1) = d(v_2) = 5$ and $d(w_1), d(w_2) \geq 4$. Then $d(v_1) + d(v_2) + d(w_1) + d(w_2) \geq 18$. Hence, there are at least 8 edges between H and K_4. Since $d_{K_4}(u_i) = 3$ and $d(u_i) \leq 5$ for $i = 1, 2, 3, 4$, there are at most 8 edges between K_4 and H.

Hence, there are exactly 8 edges between K_4 and H. Moreover, we obtain that H has exactly 5 edge, $d(w_1) = d(w_2) = 4$ and $d(u_1) = d(u_2) = d(u_3) = d(u_4) = 5$. Furthermore, each of w_1, w_2 is adjacent to exactly one vertex in $V(K_4)$.

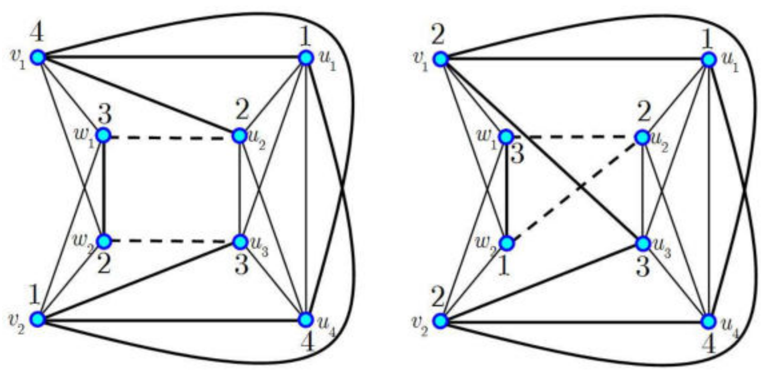

Figure 4.6: Case 2.1

If w_1 and w_2 are adjacent to different vertices in $V(K_4)$, then G must be the left graph in Figure 4.6. If w_1 and w_2 are adjacent to the same vertex in $V(K_4)$, then G must be the right graph in Figure 4.6. According to the figure, G is 4-colorable.

Case 2.2. $d(v_2) = 4$. Notice that H has at most 5 edges because H has 4 vertices and v_1, v_2 are not adjacent. Then $d_H(v_1)+d_H(v_2)+d_H(w_1)+d_H(w_2) \leq 10$. However, $d(v_1) = 5, d(v_2) = 4$ and $d(w_1), d(w_2) \geq 4$. Then $d(v_1) + d(v_2) + d(w_1) + d(w_2) \geq 17$. Hence, there are at least 7 edges between H and K_4.

Since G does not contain K_5, v_1 is not adjacent to at least one vertex from K_4, say u_4. If $d(u_4) = 5$, then the proof is done by Case 2.1 because v_1 and u_4 are nonadjacent vertices with degree 5. Suppose that $d(u_4) = 4$. Since $d_{K_4}(u_1) = d_{K_4}(u_2) = d_{K_4}(u_3) = d_{K_4}(u_4) = 3$ and $d(u_i) \leq 5$ for $i = 1, 2, 3$ and $d(u_4) = 4$ there are at most 7 edges between K_4 and H.

Hence, there are exactly 7 edges between K_4 and H. Moreover, we obtain that H has exactly 5 edge, $d(w_1) = d(w_2) = 4$ and $d(u_1) = d(u_2) = d(u_3) = 5, d(u_4) = 4$. Furthermore, each of w_1, w_2 is adjacent to exactly one vertex in $V(K_4)$.

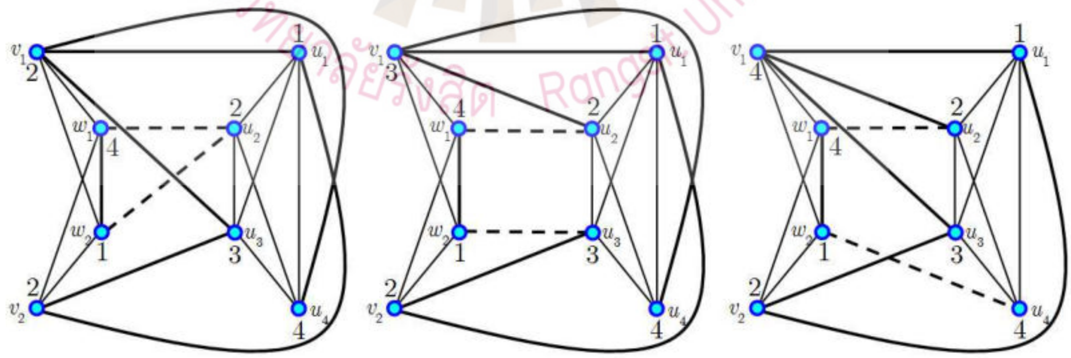

Figure 4.7: Case 2.2

If w_1 and w_2 are adjacent to the same vertex in $V(K_4)$, then G must be the left graph in Figure 4.7. If w_1 and w_2 are adjacent to different vertices in $V(K_4)$, then G must be the middle graph or the right graph in Figure 4.7. According to the figure, G is 4-colorable.

Case 3. $G-v_1-v_2$ contains neither K_4 nor $C_5 \vee K_1$. By Lemma 2.9, $G-v_1-v_2$ is 3-colorable. Hence, all vertices from $G-v_1-v_2$ are labeled by 3 colors and v_1, v_2 are labeled by the fourth color. Therefore, G is 4-colorable. □

Theorem 4.17. *If G is an 8-vertex graph having no $C_5 \vee K_2$ and K_5, then G is $(4,7)$-choosable.*

Proof. Assume that G is an 8-vertex graph having no $C_5 \vee K_2$ and K_5. Then for every vertex v, $G[N(v)]$ have no $C_5 \vee K_1$ and K_4.

Case 1. $\Delta(G) \leq 4$. Then G is 4-colorable by Theorem 2.16. Hence, G is 4-choosable by Corollary 2.15.

Case 2. $\Delta(G) = 5$. Then G is 4-colorable by Lemma 4.16. Hence, G is 4-choosable by Corollary 2.15.

Case 3. $\Delta(G) = 6$. Let v be a vertex with $d(v) = 6$ and w is the vertex which is not adjacent to v. Let L be a $(4,7)$-list assignment of G. Since $|L(v)| = |L(w)| = 4$ and $|L(V(G))| = 7$, $L(v)$ and $L(w)$ have a common color, say c. Hence, we label v and w by color c.

By Lemma 2.9, $G-w-v$ is 3-colorable because it has no $C_5 \vee K_1$ and K_4. By Corollary 2.14, $G-w-v$ is 3-choosable. That is, G is L-colorable.

Case 4. $\Delta(G) = 7$. Let v be a vertex with $d(v) = 7$. Notice that v is adjacent to the remaining vertices of G. That is, $G-v$ have no $C_5 \vee K_1$ and K_4 According to Corollary 2.12, $G-v$ is $(3,6)$-choosable. Hence, G is $(4,7)$-choosable. □

Lemma 4.18. *$K_{2,2,2}$ is L-colorable if $|L(V(K_{2,2,2}))| = 4$ and each list has size* 3 *except one list of size* 2.

Proof. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$ be partite sets of $K_{2,2,2}$. Let L be a list assignment of $K_{2,2,2}$ such that $|L(V(K_{2,2,2}))| = 4$ and each list has size 3 except one list of size 2. Without loss of generality, suppose that $|L(x_1)| = 2$ and $|L(v)| = 3$ for the remaining 5 vertices. Since $|L(V(K_{2,2,2}))| = 4$, there is a color $c \in L(x_1) \cap L(x_2)$. Then we label x_1 and x_2 by color c. Hence, each of the remaining 4 vertices has 2 available colors. Since C_4 is 2-choosable, the remaining vertices can be labeled. □

Lemma 4.19. *G_4 is $(3, 5)$-choosable.*

Proof. Let L be a $(3, 5)$-list assignment of G_4. Since $|L(V(G_4))| = 5$, we obtain that $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_4) \cup L(x_5)| \geq 4$ or $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_6) \cup L(x_7)| \geq 4$. Without lose of generality, suppose that $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_4) \cup L(x_5)| \geq 4$.

Since $|L(x_6)| = |L(x_7)| = 3$ but $|L(V(G_4))| = 5$, there is a color $c \in L(x_6) \cap L(x_7)$. Then x_6 and x_7 are labeled by color c. Let $L' = L - c$ be the list assignment of $G - x_6 - x_7$. Since $|L'(x_i)| \geq 2$ for $i = 1, 2, 3, 4, 5$ and $|\bigcup_{x \in V(G - x_6 - x_7)} L'(x)| \geq 3$, $G - x_6 - x_7$ is L'-colorable by Example 2.1. □

Lemma 4.20. *G_7 is $(3, 5)$-choosable.*

Proof. Notice that x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \mod 7$. Let L be a $(3, 5)$-list assignment of G_7.

Case 1. There is a color c that appears in exactly 1 lists. Without loss of generality, suppose $c \in L(x_1)$ but $c \notin L(v)$ for the remaining vertices v. Then we label x_1 by color c. Each of the remaining six vertices has 3 available colors. Since $G_7 - x_1$ does not contain K_4 or $C_5 \vee K_1$, it is 3-colorable by Lemma 2.9. Hence, $G_7 - x_1$ is 3-choosable by Corollary 2.14. That is, the remaining vertices can be labeled.

Case 2. There is a color c that appears in exactly 2 lists. Without loss of generality, we may prove only 3 subcases because x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \mod 7$.

Case 2.1 $c \in L(x_1) \cap L(x_2)$. Then we label x_1 and x_2 by color c. Hence, $G - x_1 - x_2$ is 3-colorable by Lemma 2.9. Consequently, it is 3-choosable by Corollary 2.14.

Case 2.2 $c \in L(x_1) \cap L(x_3)$. Then we label x_1 by color c. Lemma 4.18 confirms that the remaining 6 vertices can be labeled.

Case 2.3 $c \in L(x_1) \cap L(x_4)$. Then we label x_1 by color c. Again, Lemma 4.18 confirms that the remaining six vertices can be labeled.

Case 3. There is a color c that appears in exactly 4. According to x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \mod 7$ Suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_6)$. We label x_1 and x_2 by color c. Lemma 2.10 confirms that the remaining five vertices can be labeled.

Case 4. There is a color c that appears in exactly 5. According to x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \mod 7$, Suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_6)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5) \cap L(x_6)$. We label x_1 and x_2 by color c. Lemma 2.10 confirms that the remaining five vertices can be labeled.

Case 4. Each color appears in exactly 3 or 6 or7 lists. Let x, y, z be the number that appears in exactly $3, 6, 7$ lists, respectively. Then $x + y + z = 5$ and $3x + 6y + 7z = 21$. There is no integer solutions for this system. Hence, this case is impossible. □

Theorem 4.21. *If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4, then G is $(3, 5)$-choosable.*

Proof. Assume that G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4. Let L be a $(3, 5)$-list assignment of G. Since G have no $C_5 \vee K_1$ and K_4, G have no $C_5 \vee K_2$ and K_5. According to Lemma 2.9, G is 4-colorable. Then $\chi(G) \leq 4$. If $\chi(G) \leq 3$, we apply Corollary 2.14. Hence, G is 3-choosable. Otherwise, suppose that $\chi(G) = 4$. By Lemma 2.11, we obtain that $\delta(G) \geq 3$ and $\Delta(G) = 4$. By Lemma 2.17, G must be one of the seven graphs in Figure 2.2. By Lemma 4.19, G_1, G_2, G_4 and G_4 are $(3,5)$-choosable. By Lemma 4.20, G_5, G_6 and G_7 are $(3,5)$-choosable. □

Theorem 4.22. *Let $k \geq 3$. If an n-vertex graph G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k - 1$.*

Proof. Let G be a graph with n vertices and $k \geq 3$. Assume that G does not contain K_{k+1} and $C_5 \vee K_{k-2}$.

Let L be a (k,t)-list assignment such that $t = kn - k^2 - 2k - 1$. To apply Theorem 2.5, let $S \subseteq V(G)$ such that $|L(S)| < |S|$. We need to prove that $G[S]$ is $L|_S$-colorable.

By Lemma 2.6, $|L(S)| \geq t - (n - |S|)k = (kn - k^2 - 2k - 1) - (kn - |S|k) = |S|k - k^2 - 2k - 1$. Then $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$. It follows that $|S| < k + 3 + \frac{4}{k-1}$.

Case 1. $|S| \leq k + 2$. Then $G[S]$ is $L|_S$-colorable.by Lemma 2.7.

Case 2. $|S| = k + 3$. Then $G[S]$ is $L|_S$-colorable. by Lemma 2.8 and Lemma 2.9.

Case 3. $|S| \geq k + 4$. Then $k + 4 \leq |S| < k + 3 + \frac{4}{k-1}$. Hence, $k = 3$ or $k = 4$.

Case 3.1. $k = 3$. Then $3 + 4 \leq |S| < 3 + 3 + \frac{4}{3-1}$. Consequently, $|S| = 7$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$, we obtain that $|L(S)|$ is 5 or 6. If $|L(S)| = 6$, then $G[S]$ is $L|_S$-colorable by Corollary 2.12. If $|L(S)| = 5$, then $G[S]$ is $L|_S$-colorable by Theorem 4.21.

Case 3.2. $k = 4$. Then $4 + 4 \leq |S| < 4 + 3 + \frac{4}{4-1}$. Consequently, $|S| = 8$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$, we obtain that $|L(S)| = 7$. By Theorem 4.17, $G[S]$ is $L|_S$-colorable. □

CHAPTER V

CONCLUSIONS AND CONJECTURES

5.1 Conclusions

Thanks to (Charoenpitseri, Punnim, and Uiyyasathian, 2011) and (Ruksasakchai, and Nakprasit, 2013), a characterization of (k,t)-choosability of n-vertex graphs when $k \geq 3$ and $t \geq kn - k^2 - 2k$ is obtained. According to (Charoenpitseri, 2013), a $(2,t)$-choosability of n-vertex graphs when $t \geq 2n - 8$ is obtained. In this research subject, we give a characterization of (k,t)-choosability of n-vertex graphs when $k \geq 3$ and $t = kn - k^2 - 2k - 1$.

5.2 Conjecture

Let $k \geq 3$. If an n-vertex graph has no K_{k+1} and $C_5 \vee K_{k-2}$, then it is (k,t)-choosable of $t = kn - k^2 - 2k - 2$.

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APPENDIX

A partial result of this research report is submitted to public to International Journal of Mathematics and Mathematical Sciences.

Extended results on (k,t)-choosability of graphs

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Abstract

A (k,t)-list assignment L of a graph G is a mapping which assigns a set of size k to each vertex v of G and $|\bigcup_{v\in V(G)} L(v)| = t$. A graph G is (k,t)-choosable if G has a proper coloring f such that $f(v) \in L(v)$ for each (k,t)-list assignment L.

In 2011, Charoenpanitseri, Punnim and Uiyyasathian gave a characterization of (k,t)-choosability of n-vertex graphs when $t \geq kn - k^2 - 2k + 1$ and left open problems when $t \leq kn - k^2 - 2k$ Recently, Ruksasakchai and Nakprasit obtain the results when $t = kn - k^2 - 2k$. In this paper, we extend the results to case $t = kn - k^2 - 2k - 1$.

Keywords : list assignments, list colorings, choosable

1 Introduction

A graph G is called *k-colorable* if every vertex of G can be labeled by at most k colors and every adjacent vertices receives distinct colors. The smallest number t such that G is t-colorable is called the *chromatic number* of G, denoted by $\chi(G)$. A *k-list assignment* L of a graph G is a function which assigns a set of size k to each vertex v of G. A (k,t)*-list assignment* of a graph is a k-list assignment with $|\bigcup_{v\in V(G)} L(v)| = t$. Given a list assignment L, a proper coloring f of G is an *L-coloring* of G if $f(v)$ is chosen from $L(v)$ for each vertex v of G. A graph G is *L-colorable* if G has an L-coloring. Particularly, if L is a (k,k)-list assignment of G, then any L-coloring of G is a *k-coloring* of G. A graph G is (k,t)*-choosable* if G is L-colorable for every (k,t)-list assignment L. If a graph G is (k,t)-choosable for each positive number t then G is called *k-choosable* and the smallest number k satisfying this property is called the *list chromatic number* of G denoted by $ch(G)$.

List coloring is a well-known problem in the field of graph theory. It was first studied by Vizing[13] and by Erdős, Rubin and Taylor[4]. They give a characterization of 2-choosable graphs. For $k \geq 3$, there is no characterization of k-choosable graphs. There are only results for some classes of graphs. For example, all planar graphs are 5-choosable, while some planar graphs are 3-choosable. (See [10],[11],[12],[14],[15],[16],[17].) In order to simplify the problem, (k,t)-choosability is defined. It is a partial problem of k-choosability. Instead of proving a graph can always be colored for entire k-list assignments, we prove the graph can be colored for k-list assignments that have exactly t colors. In 2011, (k,t)-choosability of graphs was explored in [3]. They proved the following theorem.

Theorem 1.1. *[3] For an n-vertex graph G, if $t \geq kn - k^2 + 1$ then G is (k,t)-choosable.*

Moreover, they showed that the bound is best possible by proving if $t \leq kn - k^2$, then an n-vertex graph containing K_{k+1} is not (k,t)-choosable. Furthermore, they keep investigating the (k,t)-choosability to obtain another interesting theorem.

Theorem 1.2. *[3] Let $k \geq 3$. A K_{k+1}-free graph with n vertices is (k,t)-choosable for $t \geq kn - k^2 - 2k + 1$.*

Again, they showed that the bound is best possible for K_{k+1}-free graphs with n vertices by proving an n-vertex graph containing $C_5 \vee K_{k-2}$ is not (k,t)-choosable for $t \leq kn - k^2 - 2k$. In conclusion, they gave a characterization of (k,t)-choosability of n-vertex graphs when $t \geq kn - k^2 - 2k + 1$.

In 2013, Ruksasakchai and Nakprasit gave a characterization of $(k, kn-k^2-2k)$-choosability of n-vertex graphs as shown in the following theorem.

Theorem 1.3. *[9] Let G be a graph with n vertices and $k \geq 3$. If G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k$*

Results on $(2,t)$-choosability of n-vertex graphs are almost completed by [8] and [2]. Here, we focus on $k \geq 3$. We will prove that if an n-vertex graph G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k - 1$.

2 Preliminaries

Throughout the paper, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of G. For $X \subseteq V(G)$, $G - X$ is the graph obtained from deleting all vertices of X from G. In case $X = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph induced by* X, denoted by $G[X]$ is the graph obtained from deleting all vertices of $V(G)$ outside X. The notaion $d(v)$ stands for the degree of v in G. For a subgraph H of G, $d_H(v)$ stands for the degree of v in H.

Let $S \subseteq V(G)$. If L is a list assignment of G, we let $L|_S$ denote L restricted to S and $L(S)$ denote $\bigcup_{v \in S} L(v)$. For a color set A, let $L - A$ be the new list assignment obtained from L by deleting all colors in A from $L(v)$ for each $v \in V(G)$. When A has only one color a, we write $L - a$ instead of $L - \{a\}$.

Example 2.1. *The cycle C_n is $(2,t)$-choosable unless n is odd and $t = 2$.*

Note that a graph G is $(2,2)$-choosable if and only if G is 2-colorable. Hence, C_n is $(2,2)$-choosable if and only if n is even. It remains to show that all of the cycles are $(2,t)$-choosable for $t \geq 3$.

Let $t \geq 3$ and L be a $(2,t)$-list assignment of C_n. Thus there are two adjacent vertices $v_1, v_n \in V(G)$ such that $L(v_1) \neq L(v_n)$. Let $v_2, v_3 \dots, v_{n-1}$ be remaining vertices along the cycle C_n where v_i is adjacent to v_{i+1} for $i = 1, 2, \dots, n-1$. First we assign v_1 a color c in $L(v_1)$ which is not in $L(v_n)$ and then we assign vertex v_2 a color in $L(v_2)$ different from c and so on. This algorithm guarantees that each pair of adjacent vertices receives distinct colors.

Theorem 2.2 and Lemma 2.3 will be combined to obtain a result on (k,t)-choosability of graphs.

Theorem 2.2. *[5] Let L be a list assignment of a graph G and let $S \subseteq V(G)$ be a maximal non-empty subset such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$-colorable then G is L-colorable.*

Lemma 2.3. *[3] Let $A_1, A_2, \dots, A_n$ be k-sets and $J \subseteq \{1, 2, \dots, n\}$. If $|\bigcup_{i=1}^{n} A_i| \geq p$, then $|\bigcup_{i \in J} A_i| \geq p - (n - |J|)k$.*

The following statements appear in [3] and [9]. The authors apply them to obtain characterizations of (k,t)-choosability of n-vertex graphs. We also need the tools, as well.

Lemma 2.4. *[3] Let G be an n-vertex graph. If $k \geq n-2$ and G is K_{k+1}-free, then G is (k,t)-choosable for any positive integer k.*

Lemma 2.5. *[3] If a $(k+3)$-vertex graph is K_{k+1}-free, then it is (k,t)-choosable for $t \geq k+1$.*

Lemma 2.6. *[9] Let G be an n-vertex graph where $n \geq 6$. If G does not contain K_{n-2} and $C_5 \vee K_{n-5}$, then G is $(n-3)$-colorable.*

Lemma 2.7. *[9] Let G be a graph in Figure 2.1. If each list has size 2 except exactly two lists of size 3 and $|L(x_4)| = 2$ or $|L(x_5)| = 2$, then G is L-colorable.*

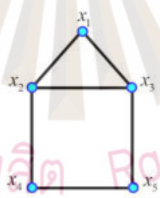

Figure 2.1: The graph for Lemma 2.7

Lemma 2.8. *[9] Let G be a graph with 7 vertices and $\chi(G) = 4$. If G does not contain K_4 and $C_5 \vee K_1$, then $\delta(G) \geq 3$ and $\Delta(G) = 4$.*

Corollary 2.9. *[9] If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4, then G is $(3,6)$-choosable.*

Recently, Ohba's conjecture is proved by Noel as shown in Theorem 2.10.

Theorem 2.10. *[6] If $|V(G)| \leq 2\chi(G) + 1$, then $ch(G) = \chi(G)$.*

The theorem is powerful because several interesting results can be obtained; for example, Corollary 2.11 and Corollary 2.12.

Corollary 2.11. *Let G be an n-vertex graph with $n \leq 7$. If G is 3-colorable, then G is 3-choosable.*

Proof. Let G be an n-vertex graph with $n \leq 7$. Assume that G is 3-colorable. Then $\chi(G) \leq 3$. Then we add some edges to G to obtain a graph H such that $\chi(H) = 3$. Since $|V(H)| \leq 7 = 2\chi(H) + 1$, we obtain that $ch(H) = \chi(H) = 3$ by Theorem 2.10. Hence G is 3-choosable because G is a subgraph of H □

Corollary 2.12. *Let G be an n-vertex graph with $n \leq 9$. If G is 4-colorable, then G is 4-choosable.*

Proof. Let G be an n-vertex graph with $n \leq 9$. Assume that G is 4-colorable. Then $\chi(G) \leq 4$. Then we add some edges to G to obtain a graph H such that $\chi(H) = 4$. Since $|V(H)| \leq 9 = 2\chi(H) + 1$, we obtain that $ch(H) = \chi(H) = 4$ by Theorem 2.10. Hence G is 4-choosable because G is a subgraph of H □

Finally, we need one more theorem and one more lemma to prove our main results.

Theorem 2.13. *[1] If G is a graph other than odd cycle and complete graph, then $\chi(G) \leq \Delta(G)$.*

Lemma 2.14. *[7] [9] Let G be a graph with $\delta(G) \geq 3$ and $\Delta(G) = 4$. If G have no K_4 and $C_5 \vee K_1$, then G must be one of the 7 graphs in Figure 2.2.*

3 Main results

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In this section, the main result is in Theorem 3.7. In order to prove the main result, we need to prove that Theorem 3.2 and Theorem 3.6. Lemma 3.1 is established for Theorem 3.2 while Lemma 3.3, Lemma 3.4 and Lemma 3.5 are established for Theorem 3.6.

Lemma 3.1. *Let G be an 8-vertex graph with $\Delta(G) = 5$. If G has no $C_5 \vee K_2$ and K_5, then G is 4-colorable.*

Proof. Let G be an 8-vertex graph with $\Delta(G) = 5$. Assume that G has no $C_5 \vee K_2$ and K_5.

If there is a vertex v such that $d(v) \leq 3$, then $G - v$ is 4-colorable by Lemma 2.6; hence, G is 4-colorable, as well. Suppose that $\delta(G) \geq 4$. Let v_1 be a vertex with $d(v_1) = 5$ of G and v_2 be a vertex which is not adjacent to v_1. Notice that $d(v_2)$ is 4 or 5.

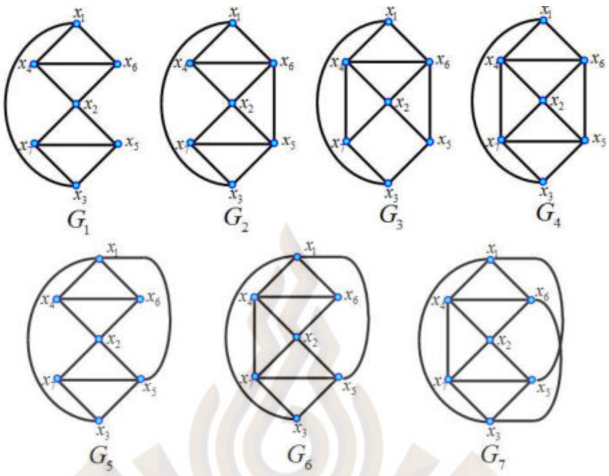

Figure 2.2: 7-vertex graph with chromatic number 4 and having no K_4 and $C_5 \vee K_1$

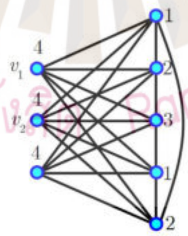

Figure 3.1: Case 1

Case 1. $G - v_1 - v_2$ contains $C_5 \vee K_1$. Under the condition $\Delta(G) = 5$, $G - v_1 - v_2$ must be $C_5 \vee K_1$. Then G must be a subgraph of the graph shown in Figure 3.1. According to the figure, G and its subgraphs are 4-colorable.

Case 2. $G - v_1 - v_2$ contains K_4. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$, $H = G - \{u_1, u_2, u_3, u_4\}$ and $V(H) = \{v_1, v_2, w_1, w_2\}$.

Case 2.1. $d(v_2) = 5$. Notice that H has at most 5 edges because H has 4 vertices and v_1, v_2 are not adjacent. Then $d_H(v_1)+d_H(v_2)+d_H(w_1)+d_H(w_2) \leq 10$. However, $d(v_1) = d(v_2) = 5$ and $d(w_1), d(w_2) \geq 4$. Then $d(v_1) + d(v_2) + d(w_1) + d(w_2) \geq 18$. Hence, there are at least 8 edges between H and K_4. Since $d_{K_4}(u_i) = 3$ and $d(u_i) \leq 5$ for $i = 1, 2, 3, 4$, there are at most 8 edges between K_4 and H.

Hence, there are exactly 8 edges between K_4 and H. Moreover, we obtain that H has exactly 5 edge, $d(w_1) = d(w_2) = 4$ and $d(u_1) = d(u_2) = d(u_3) = d(u_4) = 5$. Furthermore, each of w_1, w_2 is adjacent to exactly one vertex in $V(K_4)$.

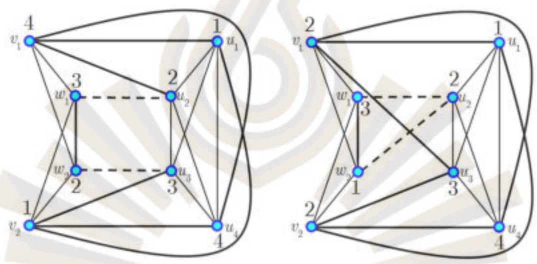

Figure 3.2: Case 2.1

If w_1 and w_2 are adjacent to different vertices in $V(K_4)$, then G must be the left graph in Figure 3.2. If w_1 and w_2 are adjacent to the same vertex in $V(K_4)$, then G must be the right graph in Figure 3.2. According to the figure, G is 4-colorable.

Case 2.2. $d(v_2) = 4$. Notice that H has at most 5 edges because H has 4 vertices and v_1, v_2 are not adjacent. Then $d_H(v_1)+d_H(v_2)+d_H(w_1)+d_H(w_2) \leq 10$. However, $d(v_1) = 5, d(v_2) = 4$ and $d(w_1), d(w_2) \geq 4$. Then $d(v_1) + d(v_2) + d(w_1) + d(w_2) \geq 17$. Hence, there are at least 7 edges between H and K_4.

Since G does not contain K_5, v_1 is not adjacent to at least one vertex from K_4, say u_4. If $d(u_4) = 5$, then the proof is done by Case 2.1 because v_1 and u_4 are nonadjacent vertices with degree 5. Suppose that $d(u_4) = 4$. Since $d_{K_4}(u_1) = d_{K_4}(u_2) = d_{K_4}(u_3) = d_{K_4}(u_4) = 3$ and $d(u_i) \leq 5$ for $i = 1, 2, 3$ and $d(u_4) = 4$ there are at most 7 edges between K_4 and H.

Hence, there are exactly 7 edges between K_4 and H. Moreover, we obtain that H has exactly 5 edge, $d(w_1) = d(w_2) = 4$ and $d(u_1) = d(u_2) = d(u_3) = 5, d(u_4) = 4$. Furthermore, each of w_1, w_2 is adjacent to exactly one vertex in $V(K_4)$.

If w_1 and w_2 are adjacent to the same vertex in $V(K_4)$, then G must be the left graph in Figure 3.3. If w_1 and w_2 are adjacent to different vertices in $V(K_4)$,

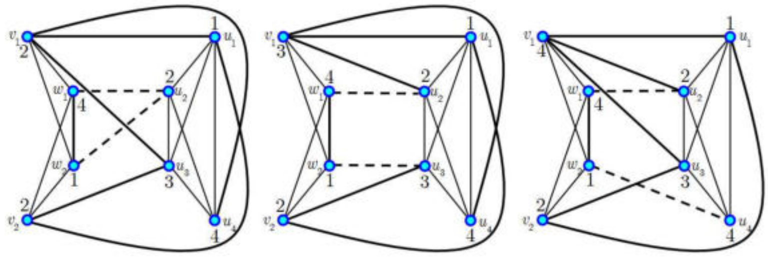

Figure 3.3: Case 2.2

then G must be the middle graph or the right graph in Figure 3.3. According to the figure, G is 4-colorable.

Case 3. $G - v_1 - v_2$ contains neither K_4 nor $C_5 \vee K_1$. By Lemma 2.6, $G - v_1 - v_2$ is 3-colorable. Hence, all vertices from $G - v_1 - v_2$ are labeled by 3 colors and v_1, v_2 are labeled by the fourth color. Therefore, G is 4-colorable. □

Theorem 3.2. *If G is an 8-vertex graph having no $C_5 \vee K_2$ and K_5, then G is $(4, 7)$-choosable.*

Proof. Assume that G is an 8-vertex graph having no $C_5 \vee K_2$ and K_5. Then for every vertex v, $G[N(v)]$ have no $C_5 \vee K_1$ and K_4.

Case 1. $\Delta(G) \leq 4$. Then G is 4-colorable by Theorem 2.13. Hence, G is 4-choosable by Corollary 2.12.

Case 2. $\Delta(G) = 5$. Then G is 4-colorable by Lemma 3.1. Hence, G is 4-choosable by Corollary 2.12.

Case 3. $\Delta(G) = 6$. Let v be a vertex with $d(v) = 6$ and w is the vertex which is not adjacent to v. Let L be a $(4, 7)$-list assignment of G. Since $|L(v)| = |L(w)| = 4$ and $|L(V(G))| = 7$, $L(v)$ and $L(w)$ have a common color, say c. Hence, we label v and w by color c.

By Lemma 2.6, $G - w - v$ is 3-colorable because it has no $C_5 \vee K_1$ and K_4. By Corollary 2.11, $G - w - v$ is 3-choosable. That is, G is L-colorable.

Case 4. $\Delta(G) = 7$. Let v be a vertex with $d(v) = 7$. Notice that v is adjacent to the remaining vertices of G. That is, $G - v$ have no $C_5 \vee K_1$ and K_4 According to Corollary 2.9, $G - v$ is $(3, 6)$-choosable. Hence, G is $(4, 7)$-choosable. □

Lemma 3.3. *$K_{2,2,2}$ is L-colorable if $|L(V(K_{2,2,2}))| = 4$ and each list has size 3 except one list of size 2.*

Proof. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$ be partite sets of $K_{2,2,2}$. Let L be a list assignment of $K_{2,2,2}$ such that $|L(V(K_{2,2,2}))| = 4$ and each list has size 3 except one list of size 2. Without loss of generality, suppose that $|L(x_1)| = 2$ and $|L(v)| = 3$ for the remaining 5 vertices. Since $|L(V(K_{2,2,2}))| = 4$, there is a color $c \in L(x_1) \cap L(x_2)$. Then we label x_1 and x_2 by color c. Hence, each of the remaining 4 vertices has 2 available colors. Since C_4 is 2-choosable, the remaining vertices can be labeled. □

Lemma 3.4. *G_4 is $(3,5)$-choosable.*

Proof. Let L be a $(3,5)$-list assignment of G_4. Since $|L(V(G_4))| = 5$, we obtain that $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_4) \cup L(x_5)| \geq 4$ or $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_6) \cup L(x_7)| \geq 4$. Without lose of generality, suppose that $|L(x_1) \cup L(x_2) \cup L(x_3) \cup L(x_4) \cup L(x_5)| \geq 4$.

Since $|L(x_6)| = |L(x_7)| = 3$ but $|L(V(G_4))| = 5$, there is a color $c \in L(x_6) \cap L(x_7)$. Then x_6 and x_7 are labeled by color c. Let $L' = L - c$ be the list assignment of $G - x_6 - x_7$. Since $|L'(x_i)| \geq 2$ for $i = 1,2,3,4,5$ and $|\bigcup_{x \in V(G-x_6-x_7)} L'(x)| \geq 3$, $G - x_6 - x_7$ is L'-colorable by Example 2.1. □

Lemma 3.5. *G_7 is $(3,5)$-choosable.*

Proof. Notice that x_i is not adjacent to x_j if and only if $i-j \equiv \pm 1 \mod 7$. Let L be a $(3,5)$-list assignment of G_7.

Case 1. There is a color c that appears in exactly 1 lists. Without loss of generality, suppose $c \in L(x_1)$ but $c \notin L(v)$ for the remaining vertices v. Then we label x_1 by color c. Each of the remaining six vertices has 3 available colors. Since $G_7 - x_1$ does not contain K_4 or $C_5 \vee K_1$, it is 3-colorable by Lemma 2.6. Hence, $G_7 - x_1$ is 3-choosable by Corollary 2.11. That is, the remaining vertices can be labeled.

Case 2. There is a color c that appears in exactly 2 lists. Without loss of generality, we may prove only 3 subcases because x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \mod 7$.

Case 2.1 $c \in L(x_1) \cap L(x_2)$. Then we label x_1 and x_2 by color c. Hence, $G - x_1 - x_2$ is 3-colorable by Lemma 2.6. Consequently, it is 3-choosable by Corollary 2.11.

Case 2.2 $c \in L(x_1) \cap L(x_3)$. Then we label x_1 by color c. Lemma 3.3 confirms that the remaining 6 vertices can be labeled.

Case 2.3 $c \in L(x_1) \cap L(x_4)$. Then we label x_1 by color c. Again, Lemma 3.3 confirms that the remaining six vertices can be labeled.

Case 3. There is a color c that appears in exactly 4. According to x_i is not adjacent to x_j if and only if $i-j \equiv \pm 1 \mod 7$ Suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_6)$. We label x_1 and x_2 by color c. Lemma 2.7 confirms that the remaining five vertices can be labeled.

Case 4. There is a color c that appears in exactly 5. According to x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 \mod 7$, Suppose that $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_5)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_6)$ or $c \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5) \cap L(x_6)$. We label x_1 and x_2 by color c. Lemma 2.7 confirms that the remaining five vertices can be labeled.

Case 4. Each color appears in exactly 3 or 6 or7 lists. Let x, y, z be the number that appears in exactly $3, 6, 7$ lists, respectively. Then $x + y + z = 5$ and $3x + 6y + 7z = 21$. There is no integer solutions for this system. Hence, this case is impossible. □

Theorem 3.6. *If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4, then G is $(3,5)$-choosable.*

Proof. Assume that G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4. Let L be a $(3,5)$-list assignment of G. Since G have no $C_5 \vee K_1$ and K_4, G have no $C_5 \vee K_2$ and K_5. According to Lemma 2.6, G is 4-colorable. Then $\chi(G) \leq 4$. If $\chi(G) \leq 3$, we apply Corollary 2.11. Hence, G is 3-choosable. Otherwise, suppose that $\chi(G) = 4$. By Lemma 2.8, we obtain that $\delta(G) \geq 3$ and $\Delta(G) = 4$. By Lemma 2.14, G must be one of the seven graphs in Figure 2.2. By Lemma 3.4, G_1, G_2, G_4 and G_4 are $(3,5)$-choosable. By Lemma 3.5, G_5, G_6 and G_7 are $(3,5)$-choosable. □

Theorem 3.7. *Let $k \geq 3$. If an n-vertex graph G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k - 1$.*

Proof. Let G be a graph with n vertices and $k \geq 3$. Assume that G does not contain K_{k+1} and $C_5 \vee K_{k-2}$.

Let L be a (k,t)-list assignment such that $t = kn - k^2 - 2k - 1$. To apply Theorem 2.2, let $S \subseteq V(G)$ such that $|L(S)| < |S|$. We need to prove that $G[S]$ is $L|_S$-colorable.

By Lemma 2.3, $|L(S)| \geq t - (n - |S|)k = (kn - k^2 - 2k - 1) - (kn - |S|k) = |S|k - k^2 - 2k - 1$. Then $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$. It follows that $|S| < k + 3 + \frac{4}{k-1}$.

Case 1. $|S| \leq k + 2$. Then $G[S]$ is $L|_S$-colorable.by Lemma 2.4.

Case 2. $|S| = k + 3$. Then $G[S]$ is $L|_S$-colorable. by Lemma 2.5 and Lemma 2.6.

Case 3. $|S| \geq k + 4$. Then $k + 4 \leq |S| < k + 3 + \frac{4}{k-1}$. Hence, $k = 3$ or $k = 4$.

Case 3.1. $k = 3$. Then $3 + 4 \leq |S| < 3 + 3 + \frac{4}{3-1}$. Consequently, $|S| = 7$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$, we obtain that $|L(S)|$ is 5 or 6. If $|L(S)| = 6$, then $G[S]$ is $L|_S$-colorable by Corollary 2.9. If $|L(S)| = 5$, then $G[S]$ is $L|_S$-colorable by Theorem 3.6.

Case 3.2. $k = 4$. Then $4 + 4 \leq |S| < 4 + 3 + \frac{4}{4-1}$. Consequently, $|S| = 8$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k - 1$, we obtain that $|L(S)| = 7$. By Theorem 3.2, $G[S]$ is $L|_S$-colorable. □

4 Open problem

Thanks to [3] and [9], a characterization of (k,t)-choosability of n-vertex graphs when $k \geq 3$ and $t \geq kn - k^2 - 2k$ is obtained. According to [2], a $(2,t)$-choosability of n-vertex graphs when $t \geq 2n - 8$ is obtained. In this paper, we give a characterization of (k,t)-choosability of n-vertex graphs when $k \geq 3$ and $t = kn - k^2 - 2k - 1$.

Conjecture Let $k \geq 3$. If an n-vertex graph has no K_{k+1} and $C_5 \vee K_{k-2}$, then it is (k,t)-choosable of $t = kn - k^2 - 2k - 2$.

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