Research Project Report

Hadwiger's Conjecture and a special case of Hadwiger's Conjecture

by

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ABSTRACT

What is the minimum number of colors required to color a map such that no two adjacent regions having the same color? Three colors are not enough to because a map with four regions with each region contacting the three other regions. However, no map have ever been found that four colors are not enough. This question first posed in the early 1850s and not solved until 1976 by Kenneth Appel and Wolfgang Haken. The *four color theorem* states that every map can be colored by using at most four colors.

For a map, we can transform it into a graph, called a *planar graph* in order to make it easier to studied and proved. According to the four color theorem, a planar graph is 4-colorable. In other words, a graph with neither K_4-minor nor $K_{3,3}$-minor is 4-colorble

Hadwiger's conjecture is a generalization of the four color theorem. Hadwiger's conjecture states that a graph with no K_{t+1}-minor is t-colorable.

In this research report, we first study the four color theorem and all results related to Hadwiger's conjecture. Then we prove that an inflation of n-graphs with $n \leq 7$ satisfying the Hadwiger's conjecture.

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CHAPTER I

INTRODUCTION

1.1 Background of the study

What is the minimum number of colors required to color a map such that no two adjacent regions having the same color? Three colors are not enough because a map with four reqions with each region contacting the three other regions require at least four colors. However, no map have ever been found that four colors are not enough. This question first posed in the early 1850s and not solved until 1976 by Kenneth Appel and Wolfgang Haken. The *four color theorem* states that every map can be colored by using at most four colors.

For a map, we can transform it into a graph, called a *planar graph* in order to make it easier to be studied and proved. According to the four color theorem, a planar graph is 4-colorable.

In 1937, Wagner [39] found a characterization of a planar graph, which states that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor. Hence, we can conclude that a graph which does not contain K_5 or $K_{3,3}$ as a minor is 4-colorble.

Hadwiger's conjecture is a generalization of the four color theorem. Hadwiger's conjecture states that a graph with no K_{t+1}-minor is t-colorable.

For case $t \leq 3$, it was proved by Hadwiger [16] when he proposed the conjecture. In 1937, Wagner [39] proved that the four color problems implies Hadwiger's conjecture when $t = 4$. After the four color problem was proved in 1976, we can conclude that the case $t = 4$ is true. Robertson, Seymour and Thomas [35] proved Hadwiger's conjecture for $t = 5$. Hadwiger's conjecture is open for $t \geq 6$.

There are interesting results related to Hadwiger's conjecture when $t \geq 6$. Kawarabayashi and Toft [20] proved that every graph contain neither K_7 nor $K_{4,4}$ as a minor is 6-colorable. Jakobsen [17], [18] proved that every graph with no $K_7^{=}$-minor is 6-colorable and every graph with no K_7^{-}-minor is 7-colorable Albar and Gonçalves proved that every graph with no K_7-minor is 8-colorable and every graph with no K_8-minor is 10-colorable. Rolek and Song [33] proved that every graph with no K_{t+1}-minor is $(2t-6)$-colorable for $t = 7, 8, 9$. They also proved every graph with no $K_8^{=}$-minor is 9-colorable and every graph with no K_8^{-}-minor is 8-colorable.

Since Hadwiger's conjecture is very dificult in general cases, it make sense to study for a special class of graphs. The conjecture was proved for line graphs by Reed and Seymour [31], power of cycles and their complement by Li and Liu [23], circular arc graphs by Belkale and Shandran [4].

Since a stronger conjecture, known as Hajós conjecture is false for all $k \geq 6$ by surprisingly simple counterexample; an infation of 5-cycle. The counterexample is not a counterexample for Hadwiger's conjecture but an inflation of some other small graph might yield a counterexample to Hadwiger's conjecture. Hence, it make sense to study Hadwiger's conjecture for an inflation of small graphs. Pedersen [29] proved that there is no counterexample to Hadwiger's conjecture can be obtained from inflating the Petersen graph. Thomassen [37] proved that a graph G is perfect if and only if every inflation of G satisfied Hajós conjecture. That is, Hajós conjecture is true for every inflation of a perfect graph. Therefore, Hadwiger's conjecture is also true for every inflation of a perfect graph. Plummer, Stiebitz and Toft stated that there is no counterexample to Hadwiger's conjecture can be obtained from inflating a graph with independence number at most 2 and order at most 11. Casselgren and Pedersen [8] prove that no counterexample to Hadwiger's conjecture can be obtained by inflating a 3-colorable graph. We can conclude that for a graph G with at most 11 vertices, If G is perfect or $\alpha(G) \leq 2$ or $\chi(G) \leq 3$, then G satisfies Hadwiger's conjecture.

In this research report, we prove that an inflation of n-graphs with $n \leq 7$ and minimum degree 3 satisfying the Hadwiger's conjecture.

1.2 Research Methodology

Assumption

Hadwiger's conjecture is true for every graph.

Objective

Prove Hadwiger's conjecture for a new special classes of graphs. We prove that an inflation of n-graphs with $n \leq 7$ and minimum degree 3 satisfying the Hadwiger's conjecture.

Methodology

1. Study all knowledge related to graph coloring in order to apply to prove a case of Hadwiger's conjecture.

2. Study all research article related to Hadwiger's conjecture.

3. Try to select a graph that might yield a counterexample.

4. If we find a counterexample, then this research article is finished and we can publish an article.

5. If we cannot find any counterexample, we will prove a new class of graphs that satisfies Hadwiger's conjecture.

Results

Find a counterexample to Hadwiger's conejecture or prove a new class of graphs satisfying the Hadwiger's conjecture.

Discussion

There is small possibility to find a counterexample to Hadwiger's conjecture. Hence, this research article wii focus on find a new class of graphs satisfying the Hadwiger's conjecture.

Suggestion

Since a stronger conjecture, known as Hajós conjecture is false for all $k \geq 6$ by surprisingly simple counterexample; an infation of 5-cycle. The counterexample is not a counterexample for Hadwiger's conjecture but an inflation of some other small graph might yield a counterexample to Hadwiger's conjecture. Hence, it make sense to study Hadwiger's conjecture for an inflation of small graphs.

Overall

1. Cgapter 1 : Background and history of this research article
2. Chapter 2 : Definition and Basic perties of graphs and colorings
3. Chapter 3 : All results related to the four colour problem
4. Chapter 4 : All reseach article conclusions of Hadwiger's conjecture
5. Chapter 5 : We prove that an inflation of n-vertex graph with $n \leq 7$ and minimum degree 3 satisfies Hadwiger's conjecture.

CHAPTER II

PRELIMINARY

2.1 Definitions and notations

Graphs

A graph G is a pair of set (V, E) where V is a finite non-empty set of elements called *vertices* and E is a finite set of elements called edges, each of which has two associated vertices. The set V and E are called the *vertex set* and the *edge set* of G, and denoted by $V(G)$ and $E(G)$. The number of vertices of G is called *order* of G and is usually denoted by n and the number of edges of G is denoted by m. A graph with only one vertex is called *trivial*.

An edge whose end points are the same vertex is a *loop* and if two or more edges are incident to the same two vertices, then they are called *multiple edges*. Unless we say otherwise, assume that all graphs have neither loops nor multiple edges.

The *complement* $\overline{G}$ of a graph G has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if the two vertices are not adjacent in G.

Adjacency and degrees

The vertices of an edge are its *endpoints* and the edge is said to *join* these vertices. If vertices v and w are endpoints of an edge e, then the edge $e = vw$ is *incident* to v and w. Two vertices that are joined by an edge are called *neighbours* and are said to be *adjacent*. If v and w are adjacent, we sometimes write $v \leftrightarrow w$, and if v and w are not adjacent we write $v \not\leftrightarrow w$. Two edges are *incident* if they share a same endpoint.

The set $N(v)$ of neighbours of a vertex v is called its *neighbourhood.* If $X \subseteq V(G)$, then $N(X)$ denotes the set of vertices that are adjacent to some vertices of X.

The *degree* of a vertex v, is denoted by $deg(v)$ or $d(v)$ is the number of its neighbours. In a non-simple graph, a loop is counted twice. A vertex of degree 0 is an *isolated vertex* and a vertex of degree 1 is a *leaf.* A graph is *regular* if all of its vertices have the same degree, and is *k-regular* if that degree is k. A 3-regular graph is sometimes called *cubic.* The maximum degree of a graph G is denoted by $\Delta(G)$ and the minimum degree of a graph G is denoted by $\delta(G)$.

An *isomorphism* between two graphs G and H is a bijection between their vertex set that preserves both adjacency and non-adjacency. The graph G and H are *isomorphic*, denoted by $G \cong H$, if there exists an isomorphism between them.

Independent sets and cliques

A set of vertices of a graph G is an *independent set* or *stable set* if no two vertices are adjacent. The *independence number* of *stability number*, denoted by $\alpha(G)$ is the size of the largest such set.

A set of vertics is a *clique* if all pairs of vertices are adjacent. The *clique number*, denoted by $\omega(G)$ is the size of a largest clique.

Walks, paths and cycles

A *walk* in a graph is a sequence of a vertices and edges, $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, in which the edges e_i joins the vertices v_{i-1} and v_i. It is always shortened to $v_0v_1, v_2, \ldots, v_k$. This walk is siad to *go from* v_0 *to* v_k or to *connect* v_0 *to* v_k. and is called a v_0, v_k-walk. The vertices v_0 and v_k are its *endpoints*; the other vertices are *internal vertices*. The *length* of a walk is its number of edges. A walk is *closed* if the first and the last vertices are the same. Some specific types of walk are the following:

- a *path* is a walk in which no vertex is repeated
- a *cycle* is a non-trivial closed walk in which no vertex is repeated except the first and the last
- a *trail* is a walk in which no edge is repeated
- a *circuit* is a non-trivial closed walk.

Connectednes and distance

A graph is *connected* if it has a path connecting each pair of vertices, and *disconnected* if there exsit two vertices such that no path connects them. A *component* of a graph is a maximal connected subgraph.

In a connected graph, the *distance* from v to w, denoted by $d(v, w)$, is the length of a shortest v, w-path. The *diameter* of a connected graph G is the greatest distance between any pair of vertices in G. If G has a cycle, the *girth* of G is the length of a shortest cycle.

A connected graph is *Eulerian* if it has a closed trail containing all of its edges; such a trail is an *Eulerian trail.* A connected graph G is Eulerian if and only if every vertex of G has even degree.

A graph with n vertices is *Hamiltonian* if it has a cycle of length n, and is *pancycle* if it has a cycle of every length from 3 to n.

Bipartite graphs and trees

If the set of vertices of a graph G can be partitioned into two non-empty subsets so that no edge joins two vertices in the same subset, then G is *bipartite*. The two subsets are called *partite sets*. A graph is bipartite if and only if it has no odd cycle.

A graph without cycles is a *forest* and a connected graph without cycles is a *tree*. The following statements are characterizaions of a tree with n vertices:

- G is connected and has no cycle
- G is connected and has $n-1$ edges
- G has no cycle and has $n-1$ edges
- G has exactly one path between any two vertices

Special graphs

We introduce some classes of graphs here:

- a *complete graph* with n vertices, denoted by K_n, is a graph whose vertices are pairwise adjacent
- a *cycle* with n vertices, denoted by C_n is a cycle of length n.
- a *path* with n vertices, denoted by P_n is a path of length $n-1$.
- a *complete bipartite graph* with partite sets of size r and s, denoted by $K_{r,s}$ is a bipartite set with partite set of size r and s, and two vertices are adjacent if and only if they are in different partite sets.

Operations on graphs

Let G and H be graphs with disjoin vertex sets $V(G) = \{v_1, v_2, \ldots, v_r\}$ and $V(H) = \{w_1, w_2, \ldots, w_s\}$

The *union graph** $G \cup H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The unoin of k graphs isomorphic to G is denoted by kG.

The *join graph** $G + H$ is obtained from $G \cup H$ by adding an edge from each vertex in G ot each vertex in H.

The *cartesian product* $G \times H$ or $G \Box H$ has vertex set $V(G) \times V(H)$ with (v_i, w_j) is adjacent to (v_h, w_k) if either v_i is adjacent to v_h and $w_j = w_k$, or $v_i = v_h$ and w_j is adjacent to w_k.

Subgraphs and minors

If G and H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, and is a *spaning subgraph* if $V(H) = V(G)$. The subgraph *induced* by $S \subseteq V(G)$ of a graph G is the subgraph H whose vertex set is S and whose edge set consists of those edges of G that join two vertices in S.

The *deletion of a vertex* v from a graph G, denoted by $G - v$, is the subgraph obtained by removing v and all of its incident edges. More generally, if the *deletion of a vertex set* S of a graph G, denoted by $G - S$,is the subgraph obtained by removing all vertices in S. Similarly, the *deletion* of an edge e from a graph G, denoted by $G - e$, is the subgraph obtained by removing the edge e from G and the *deletion of an edge set* X of a graph G, denoted by $G - X$, is the graph obtained from G by removing all edges in X.

If the edge e join vertices v and w, then the *subdivision* of e replaces e by a new vertex u and two new edges vu and uw. Two graphs are *homomorphic* if there is some graph from which each can be obtained by a sequence of subdivisions.

The *contraction* of e replaces its vertices v and w by a new vvertex u with an edge ux if v or w uis adjacent to x in G. A *minor* of G is any graph that can be obtained from G by a sequence of edge-deletion and contraction. Note that if G has a subgraph homonorephic to H, then H is a subgraph of G.

Connectivity

A vertex v in a graph G is a *cut-vertex* if $G - v$ has more conponent than G. For a connected graph, this is eqivalent to saying that $G - v$ is disconnected. A non-trivial graph is *non-separable* if it is connected and has no cut-vertex. There following statements are characterizaions of non-separable graphs:

- every two vertices of G share a cycle
- every two edges of G share a cycle
- for any three vertices u, v and w in G, there is a v, w-path that contains u.
- for any three vertices u, v and w in G, there is a v, w-path that does not contain u.

A *block* in a graph is a maximal non-separable subgraph. Each edge of a graph lies in exactly one block, while each vertex that is not an isolated vertex lies in at least one block, those that are in more than one block being cut-vertices. The basic idea of non-separable graphs has a natural generalization. A graph G is *k-connected* if the removal of fever than k-vertices always leaves a non-trivial connected graph. The following theorem can be called the Fundamental theorem of connectivity, Manger's theorem, which is first published in 1927. Paths joining the same pair of vertices are called *internally disjoint* if they have no other vertices in common.

Manger's therem (vertex version) A graph is k-connected if and only if every pair of vertices are joined by k internally disjoint paths

The *connectivity* $\kappa(G)$ of a graph G is the maximum non-negative integer k for which G is k-connected; for example, the connectivity of the complete graph K_n is $n-1$, and a graph has connectivity 0 if and only if it is trivial or disconnected.

An edge e is a *cut-edge* or *bridge* of a graph G if $G-e$ has more components than G. Note that the removal of edge cannot increase the number of components by more than one. An edge e is a cut edge if there exist vertices v and w for which e is on every v,w-path.

A graph G is *l-edge-connected* if the removal of fewer than l edges always leaves a connected graph. The following is another version of Manger's theorem.

Manger's therem (edge version) A graph is l-edge-connected if and only if every pair of vertices are joined by l edge disjoint paths.

The *edge-connectivity* $\lambda(G)$ of a graph G is the greatest non-negative integer l for which G is l-edge-connected. It is easy to see that $\lambda(G)$ cannot exceed the minimum degree of G and it is at least as large as the connectivity. That is

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

2.2 Colorings of graphs

Vertex colorings

A *coloring* of a graph G is an assignment of a color to each vertex of G so that adjacent vertices always have different colors, and G is said to be *k-colorable* if it has a coloring with k-colors. The *chromatic number* $\chi(G)$ is the smallest value of k for which H has a k-coloring.

A graph is 2-colorable if and only if it does not contain any odd cycle. However, there is no characterization of a k-colorable graph when $k \geq 3$. The following theorems are interesting useful statments.

Theorem 2.1 (Brook1941). *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Theorem 2.2. *If G is a graph with n vertices and independence number α, then*

$$\frac{n}{\alpha} \leq \chi(G) \leq n - \alpha + 1$$

Edge colorings

An *edge coloring* of a graph G is an assignment of a color to each vertex of G so that incident edges always have different colors, and G is said to be *k-edge colorable* if it has an edge coloring with k-colors. The *edge chromatic number* $\chi'(G)$ is the smallest value of k for which H has a k-edge coloring. Obviously, $\chi'(G) \geq \Delta(G)$.

Theorem 2.3. *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Theorem 2.4. *For every graph G, the edge chromatic number $\chi'(G) \leq \Delta(G) + 1$.*

total colorings

A *total coloring* of a graph G is an assignment of a color to each vertex and each edge of G so that incident elements always have different colors, and G is said to be *k-total-colorable* if it has a total coloring with k-colors. The *total chromatic number* $\chi''(G)$ is the smallest value of k for which H has a k-total coloring. Obviously, $\chi''(G) \geq \Delta(G) + 1$.

Conjecture 1 (Total coloring conjecture). For every graph G, the edge chromatic number $\chi'(G) \leq \Delta(G) + 2$.

List colorings

If v is a vertex in a graph G, then a *color list* for v is a set $L(v)$ of colors that are permitted at v. If each vertex of G has a color listm then an L-*coloring* of G is a coloring in which the color of each vertex comes from its lists. A graph is k-choosable if a graph G has an L-coloring for every list L with $|L(v)| = k$ for all vertices v. The *list chromatic numboer* or *choice number*, denoted by $\chi_l(G)$, is the minimum number k for which G is k-list colorable. It is easy to check that $\chi(G) \leq \chi_l(G)$ and $\chi_l(G) \leq \Delta(G) + 1$.

CHAPTER III

FOUR COLOR PROBLEM

3.1 History

How many differnet colors are sufficient to color the countries on a map in such a way that no two adjacent countries have the same color?

Figure 3.1: A 4-coloring of africa

Figure 3.1 shows a typical arrangement of four color map. Notice that two regions which share a single point are not consided to be adjacent. That is, both regions may have the same colors.

After coloring a wide variety of planar graph, one find that every planar graph can be colored by using four distinct colors. To become a graph, all regions are replaced by vertices and two vertices are connected by an edge if both regions are adjacent. The graph, obtained from a map, is called a *planar graph*. The four coloring conjecture states that a planar graph is 4-colorable.

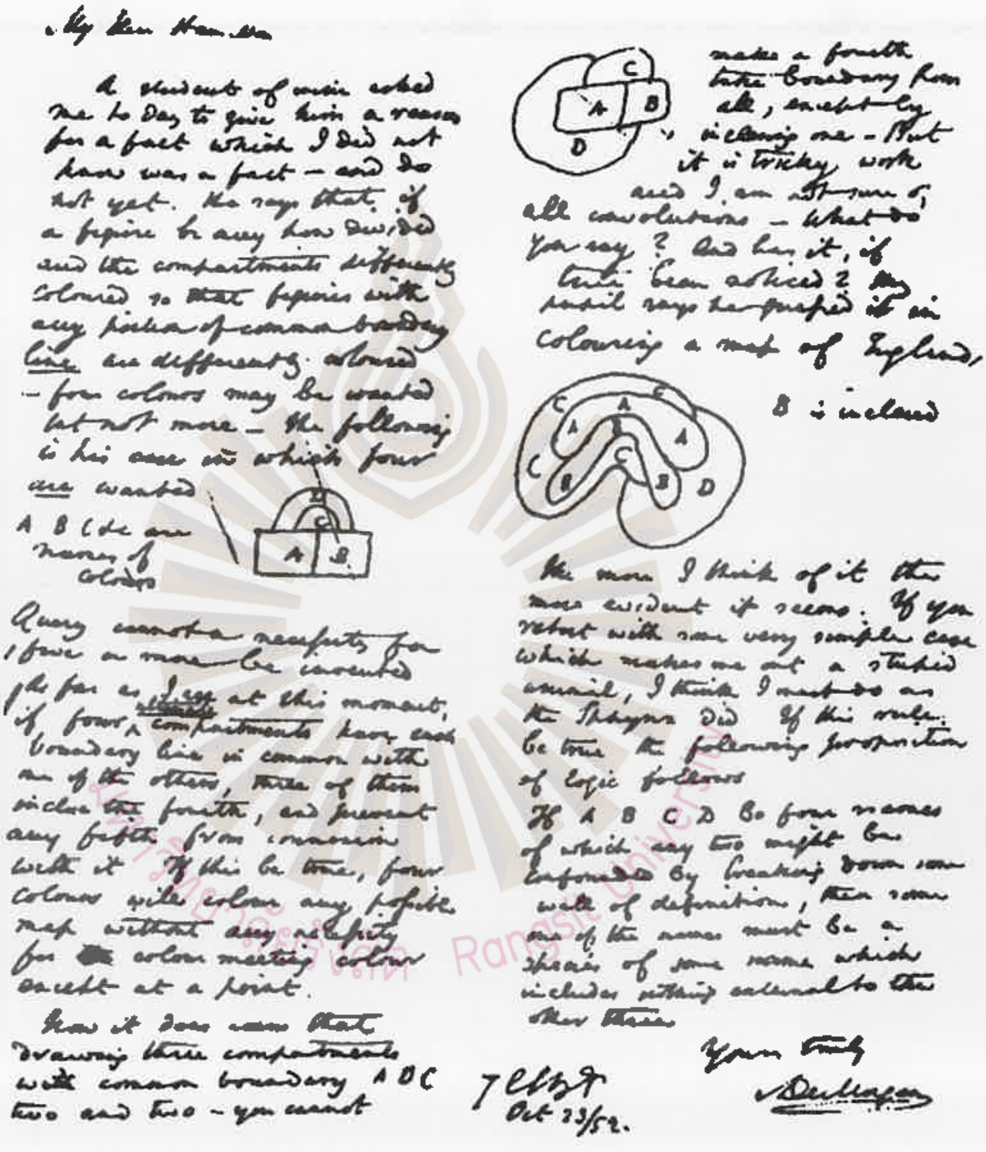
As far as is know, this conjecure was first proposed on October 23, 1852 by a young man Francis Guthrie. He discovered the statement while trying to color countries of England. At the time, Guthrie's brother, Frederick, was a student of Augustus De Morgan at University college of London. Francis inquired with Frederick, who then took it to De Morgan.

> A student of mine [Guthrie] asked me today to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be any how divided and the compartments differently colored so that figures with any portion of common boundary line are differently colored—four colors may be wanted but not more.

De Morgan too was unable to prove the conjecture. When he recognize the difficulty of the problem, he wrote to Sir William Roman Hamilton to ask for help. Hamilton immediately wrote back that he did not believe he would solve the conjecture any time soon.

The conjecture was first publshed on June 10, 1854 in British magazin, The Athenaeum. by F.G., perhaps one of the two Guthries and De Morgan posed the problem again in the same magazine in 1860. The first formal print is made by Cayley in 1879 which gave credit back to De Morgan.

In the same year, the first proof was given by Alfred Kempe; its incorrectness was pointed out by Percy Heawood, 11 years later. Another failed proof by Peter Guthrie Tait in 1880; its gap was shown by Julius Petersen in 1991. In addition to show a gap in Kempe's proof, Heawood prove *five color theorem* and generalized the four color conjecture to surfaces of arbitrary genus. Both failed proofs did have some value. Kempe discovered what became known as Kempe chains and Tait found an equivalent formulation of the Four Color Theorem in terms of 3-edge-coloring. Tait also showed that the four color theorem is equivalent to the statement that *snark* must be non-planar.



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented

As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

7 C St
Oct 23/52.

Figure 3.2: Letter of De Morgan to Hamilton, 23 Oct. 1852

The next major contribution was from Birkhoff whose work allowed Franklin in 1922 to prove that the four color conjecture is true for maps with at most 25 regions.

The main contribution came from A German mathematician, Heinrich Heesch, who developed two main concepts for the final proof; reducibility and discharging. He also expanded the concepts with Ken Durre, developed a computer test for it. Unfortunately, he was unable to procure the necessary supercomputer time to continue his work.

Finally, after the problem is open for a hundred years before it was proved in 1976 by Kenneth Appel and Wolfgang Haken [2], [3]. Initially, their proof was not accepted by all mathematician because it was proved by using computer and it cannot proved all cases by hand.

Theorem 3.1 (Four color theorem). *Every planar graph is 4-colorable.*

In 1943, Hugo Hadwiger introduced a generalization of the four color theorem which is still unsolved. [16].

Conjecture 2 (Hadwiger's conjecture). For every integer $t \geq 0$, every graph with no K_{t+1}-minor is t-colorable.

3.2 Kempe's approch

Kempe published his proof of the four color theorem in 1879 and it was accepted as a valid proof for a decade before a mistake was found. However, his proof contains several clever ideas. His ideas are also used in the complete proof of the four color problem. He proved four color problem by induction on the number of countries. On the induction step, a theorem obtained from Euler's formula can confirm that a country with adjecent to at most five countries can be found in every map. By Basic step, the map without this country is four color map. If the country is adjacent to at most three countries, there is always a remaining color for this country. If the country is adjacent to four or five countries, Kempe try to prove that a map from Basic step can re-colored in such a way that three colors are used for these adjacent countries. However, there is an error for the case five countries.

Although Kempe's argurment is fallacious, we give him credit for several clever ideas such as the inductive statement and Kempe chain argument. His ideas are essential parts for the complete proof of the four coloring problem nearly 100 years later.

It is not easy to find a political map in the real world to show Kempe's idea because countries do not normally meet in groups of more than three. Therefore, quadripoint are excepttional and rare. There are 195 countries in the world today. Taiwan is not included because the United Nations considers it represented by people's Republic of China. There are 176 international tripoints but there is only possible one international quadripoints.

Figure 3.3: the possible international quadripoint

The possible quadripoint is the borders of Namibia, Botswana, Zambia, and Zimbabwe. Actually, there is a controversy that it is not a quadripoint but two separate tripoints, some 100 or 150 meters apart. However, on the map we should consider this point as a quidripoint. That is, these four countries do not require distinct colors.

Here, we define a *Kempe chain* to be the largest set of countries you can get to from a given place by keeping to countries of a particular two colors, and crossing at edges, not vertices.

Figure 3.4: Red-Blue chain containing Botswana

If we want to re-color the africa map in such a way that three colors are used for coutries around the quadripoint, we can swap red and blue in the red-blue chain on Botswana. Since Botswana and Zambia are in distinct red-blue chain, the color on Zambia is still red.

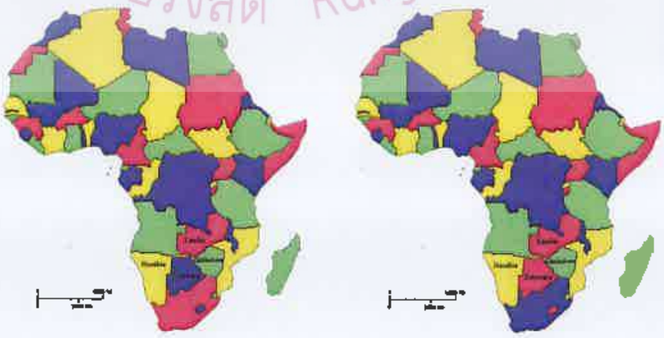


Figure 3.5: Swap red and blue in the Red-Blue chain containing Botswana

In the case that Botswana and Zambia are in the same red-blue chain, Swaping red and blue in this chain do not reduce the number of colors around the quadripoint. Fortunately, the red-blue chain separates Namibia and Zimbabwe into distinct yellow-green chains. Hence, we can swap yellow and green on the yellow-green chain containing Namibia.

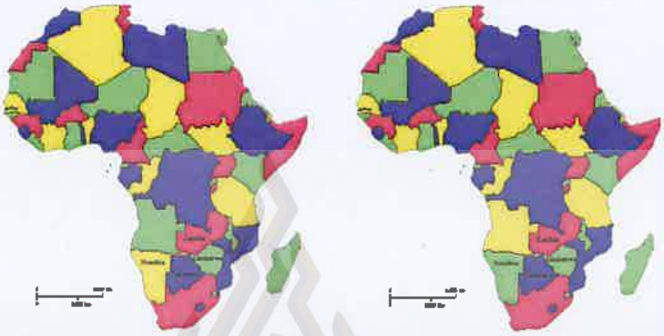


Figure 3.6: Swap yellow and green in the yellow-green chain containing Namibia

For officaial proof, see the following lemma.

Lemma 3.1. *Let M be a four colorable map. If four countries meet at a point v, then the map can be 4-colored in such a way that only three colors are used for these four countries.*

Proof. Let M be a four colorable map and let A, B, C, D be four countries that meet at a point in clockwise order. Suppose that the A, B, C and D are colored by red, green, blue and yellow, respectively.

Case 1. If A and C do not belong to the same red-blue chain, then we swap red and blue in the chain containing A.

Case 2. If A and C belong to the same red-blue chain, then B and D are seperated by this red-blue chain. Hence, we swap green and yellow in the chain containing B. □

Next, we will consider a four color map that five countries meet at a point. In the real world, there is no point that five countries meet. There exists five cities that meet at a single point; for example, five cities in Florida where they meet in the middle of Lake Okeechobee. However, the map of Florida is quite large. Then we will raise an example.

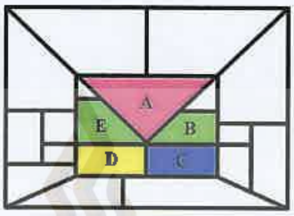


Figure 3.7: A map containing five regions that meet at a single point

Suppose that the five cities are colored by four colors as shown in Figures 3.7. We will try to color all cities in this four color map in such a way that the five cities are colored by only three colors by using Kempe chain.

If the red and blue cities do not belongs to the same red-blue chain, then we swap red and blue in the chain containing region A. Then five regions can be colored by using only three colors.

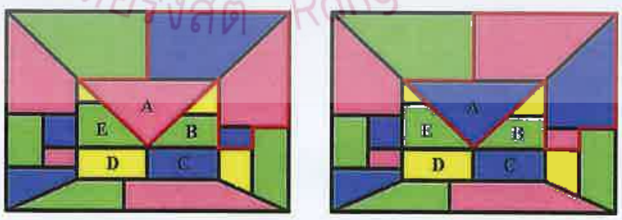


Figure 3.8: Swap the red-blue chain containing A

If the red and yellow cities do not belongs to the same red-yellow chain, then we swap red and yellow in the chain containing region A. Then five regions can be colored by using only three colors.

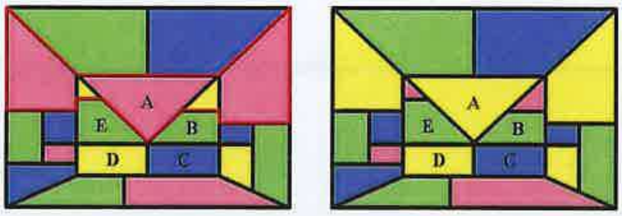


Figure 3.9: Swap the red-yellow chain containing A

For the remaining case, suppose the red and the blue cities belongs to the same red-blue chain and the red and the yellow cities belongs to the same red-green chain. Then we swap green and blue in the green-blue chain containing region E and swap green and yellow in the green-yellow chain containing region B. Hence, the five cities can be colored by using only three colors.

Figure 3.10: Swap the green-blue chain containing E and swap the green-yellow chain containing B

The proof is missing a case when the red-blue chain (A-C) and the red-green chain (A-D) have an intersection. See Figure 3.11. The same algorithm will not be working. The five cities are still be colored by four colors because whenever we swap.

Even Kempe's argument was fallacious, we must give hime credit for crever

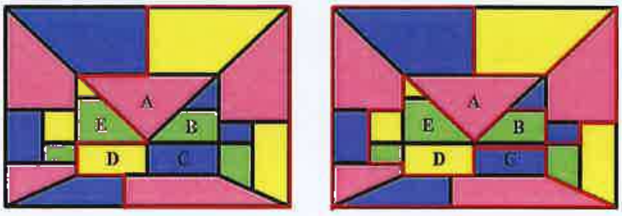


Figure 3.11: An intersection between the red-blue chain (A-C) and the red-green chain (A-D)

ideas. The inductive argument and Kempe-chain argument are essential for final proof of the four color problem and it can be applied to prove the five color theorem.

Lemma 3.2. *Let M be a five colorable map. If four countries meet at a point v, then the map can be 5-colored in such a way that only four colors are used for these five countries.*

Proof. Suppose that the five countries are colored by red, green, blue, yellow, orange in clockwise order. If red and blue cities are not in the same red-blue chain, then we can swap colors in one of the chains to obtain the desired coloring. If red and blue citues are in the same red-blue chain, then the chain separates the green and yellow cities from each other; hence, we swap colors in the green-yellow chain containing either the green city or the yellow city to obtain the disired coloring. □

Theorem 3.2. *Every plane map can be colored by at most five colors.*

Proof. We will prove by induction on the number of regions.

It is trivial for basic step when the number of regions is at most five. For induction step, suppose that every plane map with k region can be colored by at most five colors. Let M be a map with $k+1$ regions. There exists a region, say region A, that adjacent to at most five other regions because a planar graph has a vertex with degree at most five. Then we remove the region to optain a map with k regions. By induction hypothesis, the map without region A can be colored with at most five colors. If all regions adjacent to region A are colored by at exactly five colors, then we can recolor to obtain that the adjacent region are color by four colors by Lemma 3.2. That is, there is an available color for region A. □

3.3 Planar graphs

For a map, we can transform it into a graph in order to make it easier to study and proof any theorem by the following steps.

1. Drawing a vertex in each rigion
2. Drawing an edge for each pair of connected regions.

Note that two regions are siad to be connected only if their boarders are connected longer than a single point.

Figure 3.12: A graph obtained from a map

The graph we obtain from the procedure is called a *planar* graph. In other words, a *planar graph* is a graph that can be embedded in the plane. Then we obtain another version of four color problem which is easier to study.

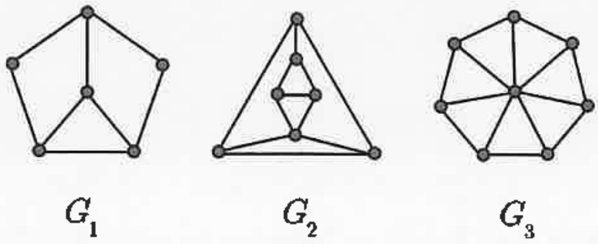


Figure 3.13: Examples of planar graph

Conjecture 3. Every planar graph is 4-colorable.

An interesting theorem is *Euler's formula* which is widely used in statements related to planar graphs.

Theorem 3.3. *If G is a planar graph with v vertices, e edges and f faces, then* $v - e + f = 2$.

Proof. We will prove by induction on the number of edges. Assume that the statement is true for a connected planar graph with v vertices, $e - 1$ edges and f faces. Let G be a planar graph with v vertices, $e1$ edges and f faces.

If G is a tree, then $v - e + f = v - (e - 1) + 1 = 2$. Suppose that G is not a tree. Hence, G contain a cycle. Let x be an edge in C. Then $G - x$ is still connected. By indcution hypothesis in $G - x$, we obtain $v - (e - 1) + f - 1 = 2$. Hence, in G, we have $v - e + f = 2$. □

Example 3.3. See Figure 3.13. We will show how to count the number of vertices, edges and faces of each graphs.

1. G_1 has 6 vertices, 8 edges and 4 faces includeing the outside face. Then $v - e + f = 6 - 8 + 4 = 2$.

2. G_2 has 7 vertices, 11 edges and 6 faces includeing the outside face. Then $v - e + f = 7 - 11 + 6 = 2$.

3. G_3 has 8 vertices, 14 edges and 8 faces includeing the outside face. Then $v - e + f = 8 - 14 + 8 = 2$.

Lemma 3.4. *If G is a planar graph with all vertices of degree at least 3, then*

$$\sum_v (6 - d(v)) \geq 12$$

Proof. Let p, q, r be the number of vertices, edges and faces of G. Assume that G is a planar graph with all vertices of degree at least 3. Then $\sum_v d(v) \geq 3p$; hence, $6p - 4q \leq 0$. Hence, $\sum_v (6 - d(v)) = 6p - \sum_v d(v) = 6p - 2q \geq (6p - 4q) + (6r - 2q) = 6p - 6q + 6r = 12$ by Euler's formula. □

Theorem 3.4. *If G is a planar graph, then G has a vertex with degree at most 5.*

Proof. Suppose that all vertices of G has degree at least 6. Then $\sum_v (6 - d(v)) \leq 0$. It is a contradiction to Lemma 3.4. □

Theorem 3.5. *If G is a planar graph with no vertex with degree less than 5, then G has at least 12 vertices with degree 5.*

Proof. It follows from Lemma 3.4. □

A planar graph G is called *triangulated* (also called maximal planar) if the addition of any edge to G results in a nonplanar graph. If a planar graph is not triangulated, then we can add an edge to the graph by without adding new vertex to obtain a planar graph with one more edge. When we keep doing, we will have a triangulated graph containing the original graph as a subgraph. Then if the triangulated graph is 4-colorable, so is the original graph. From now on, we assume that every planar graph is triangulated.

Lemma 3.5. *Let v be a vertex with degree at most 4 of G. If $G - v$ is 4-colorable, so is G.*

Proof. Let v is a vertex with degree at most 4 of G. Suppose that $G - v$ has a 4-coloring. If $d(v) \leq 3$, then there is always an available color for v. Suppose that $d(v) = 4$ and a, b, c, d are neighbors of v If G is not triangulated, then we do not know structure of edges among a, b, c, d; hence, we cannot apply Kempe's Lemma. Suppose that G is triangulated. Without loss of generality, we can say that a, b, c, d are fouvertices of a face of $G - v$. Hence, we can apply Lemma to obtain that there exists a 4-coloring of $G - v$ such that a, b, c, d use at most three colors. Hence, there is an available color for v. Therefore, the 4-coloring can be extended to G. □

Theorem 3.6. *If G is a minimal counterexample to the four color problem, then G has no vertex with at most 4.*

3.4 Reducibility and unavoidable sets

All attempts to prove the four color problem are not mainly different from Kempe in his paper. Everone try to prove by induction on the number of vertices. Even in the final proof, Appel and Haken [2], [3] also used induction on the number of vertices. The basic step starts from a graph with at most 4 vertices. The inductive step is to remove a vertex, or a set of vertices, to obtain a smaller graph. Hence, the smaller graph is 4-colorable by induction hypothesis. The remaining proof is to extend the 4-coloring to the extra vertex, or the set of extra vertices. To extend the coloring, we normally need to change the coloring so that there are available color for the set of extra vertices.

Let H be a graph. We say that *any graph contains H as a subgraph is reducible* or H *is reducible* if we can always extend a 4-coloring of $G - H$ to a 4-coloring of G.

By Lemma 3.5, we say that a vertex with degree at most 4 is reducible. As usual, we assume that there is no vertex with degree less than 5.

A *reducible configuration* is a subgraph that cannot occur in a minimal counterexample. If a planar graph contains a reducible configuration, then the planar graph can be reduced to a smaller graph. This smaller graph has the condition that if it can be colored with four colors, then the original graph can also. This implies that if the original graph cannot be colored with four colors the smaller map cannot either and so the original graph is not minimal.

Theorem 3.7. *If G is a minimal counterexample to the four color problem, then G has no seperated triangle.*

Proof. Assume that G has a seperate triangle. Then G consists of two smaller graphs which intersecte only in the triangle. If two smaller graphs are 4-colorable, then we can rename the colorings the triangle of a smaller graphs to obtain a 4-coloring of G. Hence, G is not minimal; a contradiction. □

Hence, we can say that a planar graph with a separated triangle is reducible.

An *unavoidable set* is a set of configurations such that every map that satisfies some necessary conditions for being a minimal non-4-colorable triangulation (such as having minimum degree 5) must have at least one configuration from this set.

For example, we can say that *a vertex with degree five is unavoidable.* If we CAN prove that a vertex with degree 5 is reducible, then we complete the proof of the four color problem However, it is not that easy. Kempe try to prove a graph with a vertex with degree 5 is reducible by using a chain but his proof has a false when he swaps colors of the second chain.

To prove the four color problem, we need to find a complicated unavoidable set, then prove that the unavoidable set is reducible. Historically, many mathematicians try both to find an unavoidable set and to find more a reducible configulation.

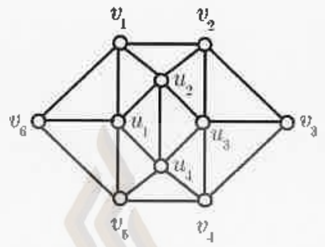


Figure 3.14: Birkhoff diamond

Another examaple of a reducible configulation is *Birkhoff diamond* which is a vertex with degree 5 with three condecutive neighbours of degree 5. Recall that our graphs are always triangulated and has minimum degree 5.

Theorem 3.8. *The Birkhoff diamond is reducible.*

Proof. Let G be a graph containing Birkhoff diamond. We will prove that G is not a minimal couterexample to the four color problem. We create a new graph from G by deleting u_1, u_2, u_3, u_4, then identifying v_2 and v_4, and joining this vertex to v_6. Suppose that the new graph is 4-colorable. Without loss of generality, suppose

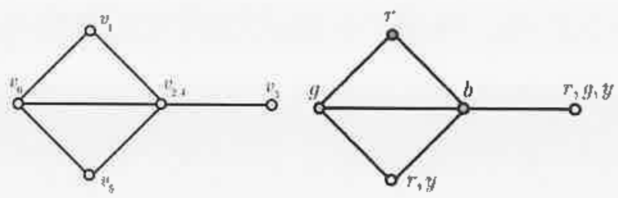


Figure 3.15: A graph obtaining from Birkhoff diamond

that $v_1, v_{2,4}, v_6$ are colored by red, blue, green, respectively. Then v_5 is colored by red or yellow, and v_3 is colored by red, green or yellow. To extend the 4-coloring to the original graph, we color u_1, u_2, u_4 by blue, green, green, respectively. Vertex

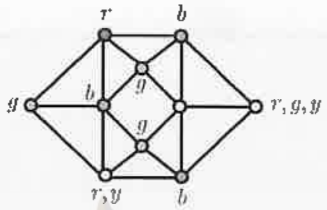


Figure 3.16: A 4-coloring

u_3 can be colored by either red or yellow depeneding a color on v_3.

Hence, G has a 4-coloring. That is, G is not a minimal counterexample. Therefore, the Birkhoff diamond is reducible. □

Lemma 3.6. *Let G be the minimal counterexample to the four color problem. Then any minimal disconnecting set in G induces at least a cycle.*

Theorem 3.9. *If G is a minimal counterexample to the four color problem, then G is 5-connected.*

Theorem 3.10. *If G is any planar graph which has a separating circuit C of length 5, such that both interior and exterior of C contain at least two vertices of G, then G is reducible.*

Theorem 3.11 (Franklin). *The configuration of a vertex of degree 5 with three consecutive neighbors of degree 5 is reducible.*

Example 3.7. Consider the configuration of a vertex of degree 5 with three neighbor of degree 5, one of degree 6 and one with arbitrary degree. Prove that this configuration is reducible.

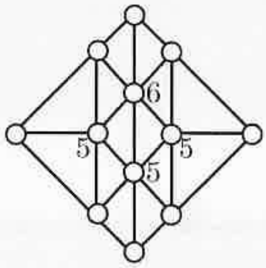


Figure 3.17: The configuration of a vertex with degree 6 with three consecutive neighbours of degree 5

Proof. Use Theorem of Birkhoff and Franklin. □

Example 3.8. Consider the configuration of a vertex of degree 5 with two consecutive neighbours of degree 5 and three neighbors of degree 6. Prove that this configuration is reducible.

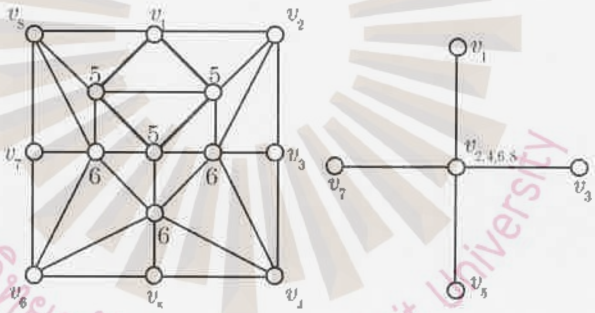


Figure 3.18: a vertex of degree 5 with two consecutive neighbours of degree 5 and three neighbors of degree 6

Example 3.9. A vertex with degree 5 with two neighbours of degree 5 and three neighbours of degree 6 is reducible.

Example 3.10. Consider the configuration of a vertex with degree 8 with five consecutive neighbours of degree 5. Prove that this configuration is reducible.

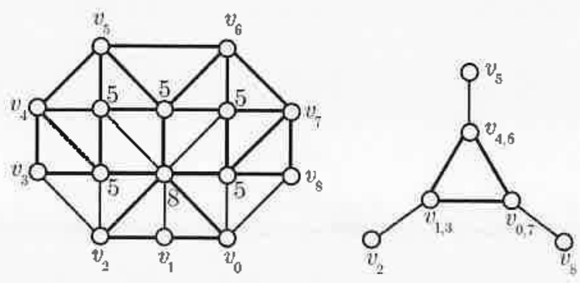


Figure 3.19: a vertex with degree 8 with five consecutive neighbours of degree 5

3.5 Discharging

For this entire section, suppose that every graph is traianglulate (i.e. maximal planar graph) and minimum degree is 5. Recall that

Proposition 3.11. $\sum_{v \in V(G)}(6 - d(v)) = 12$.

Theorem 3.12 (Wernicke 1904). *In any minimal couterexample to the four coloring problem, there is a vertex of degree* 5 *with a neighbor of degree* 5 *of* 6.

Proof. Let G be a minimal counterexample. Assume that no vertex with degree 5 is adjacent to any vertex with degree 5 or 6. Let p_i be the number of vertices with degree i and r be the number of faces. We will count the number of faces of G.

Each vertex with degree 5 is surrounded by 5 faces. Since each face is incident to a unique vertex with degree 5. Then the number of faces incident to a vertex with degree 5 is $5p_5$. Each vertex with degree 6 is surrounded by 6 faces. Since each face is incident to at most three vertex with degree 6. Then each vertex with degree 6 contribute to at least two faces. Then the number of faces incident to a vertex with degree 6 is at least $2p_6$. Then $r \geq 5p_5 + 2p_6 \geq \sum_{i=5}^{\infty}(20 - 3i)p_i = 20 - 3\sum_{i=5}^{\infty} ip_i$.

By the handshaking lemma implies $\sum_{i=5}^{\infty} ip_i = 2q = 3r$. Then $20q - 21r =$

$10\sum_{i=5}^{\infty} ip_i - 7\sum_{i=5}^{\infty} ip_i = 3\sum_{i=5}^{\infty} ip_i$. Combining the two inequality to obtain that $r \geq 20p - 20q + 21r = r + 40$ by Euler's formula, obtaining a contradiction. □

Theorem 3.13 (Franklin, 1923). *In any minimal counterexample to the four color problem, there is a vertex with degree* 5 *with two neighbours, each of degree* 5 *or* 6.

To find a complicated unavoidable set, we need to consider not only the neighbours of a vertex of degree 5 but also the neighbors of the neighbors. The counting argument is more difficult. The idea of *discharging*, introduced by Heesch, can overcome this difficulty.

We put a *charge* of $6 - d(v)$ on each vertex v. Recall that our graph has minimum degree 5. The the vertices with degree 5 get a charge of $+1$, the vertices with degree 6 are uncharged and all remaining vertices have a negative charge. By Lemma 3.4, the total charge of a graph is always 12. We first assume that there exists a graph that contains non of the configuration in an unavoidable set. Then we will redistribute the charge in some way to obtain the total charge to be 0 or negative. According to charge conservation, we obtain a contradiction. That is, every graph contains at least a configuration from the unavoidable set.

Theorem 3.14. *The set U_1 in Figure 3.20 is an unavoidable set.*

Figure 3.20: An unavoidable set U_1

Proof. Recall that our graph has minimum degree 5 and triangulated. Assume that there exsits a graph G which has non of the configuration in U_1. Then all neighbors of all vertices of degree 5 has degree 7. Then we redistribute charge by the following rule.

Every vertex with degree 5 gives a charge of $\frac{1}{5}$ to each of its neighbours.

Vertices with degree 6 are not affected by the rule because there is no pair of vertices of degree 5 and 6. After the redistribution, all vertices with degree 5 have a charge of 0. Let v be a vertex with degree $k \geq 7$. Then v has no two consecutive neighbors of degree 5 because there is no pair of vertices of degree 5. That is, v has at most $\frac{k}{2}$ neighbors of degree 5. Hence, v acquires at most $\frac{1}{5} \cdot \frac{k}{2}$. The v has a charge of at most $6 - k + \frac{k}{10} = 6 - \frac{9}{10} \cdot k \leq 6 - \frac{9}{10} \cdot 7 \leq 0$. Therefor, ther total charge of G is negative, a contradiction. □

Theorem 3.15. *The set U_2 in Figure 3.21 is an unavoidable set.*

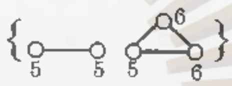


Figure 3.21: An unavoidable set U_2

Proof. Recall that our graph has minimum degree 5 and triangulated. Assume that there exsits a graph G which has non of the configuration in U_2. Then vertices of degree 5 has no neighbors with degree 5 and has no consecutive neighbors of degree 6. Consequently, the vertices of degree 5 has at most two neighbors of degree 6. Hence, the vertices of degree 5 has at least three neighbors of degree at least 7. Then we redistribute charge by the following rule.

Every vertex with degree 5 gives a charge of $\frac{1}{3}$ to each of its neighbours of degree at least 7.

Vertices with degree 6 are not affected by the rule because there is no pair of vertices of degree 5 and 6. After the redistribution, all vertices with degree 5 have a nonpositive charge. All vertices with degree 7 has at most three neighbors of degree 5. Hence, its charge is at most $(7-6)+3\cdot\frac{1}{3}=0$. Let v be a vertex with degree $k\geq 8$. Then v has at most $\frac{k}{2}$ neighbors of degree 5. Hence, v acquires at most $\frac{1}{3}\cdot\frac{k}{2}$. The v has a charge of at most $6-k+\frac{k}{6}=6-\frac{5}{6}\cdot k\leq 6-\frac{5}{6}\cdot 8<0$. Therefor, ther total charge of G is negative, a contradiction. □

To prove the four color problem, it requires a sufficiently complicated discharging rule and a sufficiently sophisticated unavoidable set. Then we need to prove that every graph contains a configuration from the unavoidable set by the discharging rule. Finally, we need to prove that every graph in the unavoidable set is reducible.

In the final prove, Appel and Haken [2] [3] used a discharging algorithm made up from over 300 separate rules, befored Robertson et. al. found a simpler algorithm of just 32 rules.

CHAPTER IV

Hadwiger's Conjecture

4.1 Minor and subdivision

Since Hadwiger's conjecture state that every graph with no K_{t+1}-minor is t-colorable for every integer $t \geq 0$. Then we need to study topics realated to minor.

The *contraction* of e replaces its vertices v and w by a new vertex u with an edge ux if v or w uis adjacent to x in G. We denote the graph obtained this way by $G \backslash vw$.

Figure 4.1: a graph G and its minor

A *minor* of G is any graph that can be obtained from G by repeatedly deleting vertices and edges and contracting edges. We say that G *contains* H as a minor and write $G \geq H$. It is easy to see that the minor relation is transitive. That is, if $G \geq H$ and $H \geq F$, then $G \geq F$.

A *subdivision* of a graph G is a graph obtained from G by replacing some of its edges by internally vertex disjoint paths.

Lemma 4.1. *If a subdivision of H is a subgraph of G, then $H \leq G$.*

Proof. Assume that there exists a subdivision of H is a subgraph of G, say H_0. Since H_0 is a subgraph of G, the graph H_0 can be obtained from G by deleting edges and vertices. Since H_0 is a subdivision of H, the graph H can be obtained from H_0 by edge contraction. □

The converse of Lemma 4.1 is not true. See Figure 4.1, H can be obtained from G by contraction vw. However, any subdivision of H cannot be a subgraph of G because H has a vertex of degree 6 but the maximum degree of G is 4.

An *H-model* in a graph G is a collection $\{S_x : x \in V(H)\}$ of pairwise vertex-disjoint connected subgraphs of G (called branch sets) such that, for every edge $xy \in E(H)$, some edge in G joins a vertex in S_x to S_y.

Lemma 4.2. *A graph G contains an H-model if and only if H is a minor of G.*

Proof. Let H and G be graphs. Suppose that G contains an H-model. We will prove that H can be obtained from G by deleting vertices and edges and contracting edges. For each connected subgraph of G from H-model, we repeat contracting until obtaining a single vertex. Then we delete the remaining vertices and edges which do not belong to H. By definition of H-model, we obtain all vertices of H and all edges of H.

In the opposite direction, assume that H is a minor of G. We will show that there exists an H-model of G by induction on $|V(G)|$. Since H is a minor of G, there is a sequence of vertex deletion, edge deletion and edge contraction to obtain H. If a subgraph of G has an H-model, so is G. Hence, we may assume that the first operation is contraction on edge vw. Let u be the new vertex in $G' = G\backslash vw$ obtained from identifying v and w.

By the induction hypothesis, there exsits an H-model of G', say μ'. Then we constuct a new H-model of G by deleting u and adding vertices u, v and edges uv. □

Lemma 4.3. *Let H be a graph with maximum degree 3. A graph G contains H as a minor if and only if a subdivision of H is a subgraph of $G.H_d \cap P = \varnothing$.*

Proof. Let H be a graph with maximum degree 3. By Lemma 4.1, if a subdivision of H is a subgraph of G, then G contains H as a minor. In the opposite direction, assume that G contains H as a minor. Notice that if G has a subgraph such that a subdivision of H is a subgraph of G', then the subdivision is also a subgraph of G. Without loss of generality, suppose that no proper subgraph of G contains H as minor. Let $\mu = S_1, S_2, \ldots, S_{V(H)}$ be an H-model in G. Recall that S_i is a connected subgraph G.

Accordding the fact that no proper subgraph of G contains H as minor, we obtain the following statement.

1. Each S_i is a tree; otherwise, we can an edge from S_i and obtain a proper subgraph of G containing H as a minor.

2. Each leaf of S_i is connected to others S_j; otherwise, we can delete the leaf and obtain a proper subgraph of G containing H as a minor.

3. There is exactly one edge between the pair of S_i, S_j; otherwise, we can delete an edge and obtain a proper subgraph of G containing H as a minor.

Then each S_i is either K_1 or subdivision of K_2 or subdivision of $K_{1,3}$. Hence, from a vertex of H, we can construct G by the following operations.

1. If S_v is K_1, then we do nothing.

2. If S_v is a subdivision of K_2, we first subdivide v with an adjacent vertex to obtain K_2. Then we can subdivide to obtain S_v.

3. If S_v is a subdivision of K_3, we first subdivide v with three adjacent vertices to obtain $K_{1,3}$. Then we can subdivide to obtain S_v.

□

Theorem 4.1. *[13] Let G be a k-chomatic critical graph. If $S = \{v_1, v_2\}$ is a separated set of $V(G)$, then $d(v_1) + d(v_2) \geq 3k - 5$.*

Proof. Let G be a k-chomatic critical graph. Assume that $S = \{v_1, v_2\}$ is a separated set of $V(G)$. Then $G - \{v_1, v_2\}$ has at least two components, say $C_1, C_2, \ldots, C_n$. Define H_i for $i = 1, 2, \ldots, n$ be a subgraph of G obtained from deleting all components except C_i from G.

Then H_i for all i is $(k-1)$-colorable since G is critical. Moreover, there is no edge between v_1 and v_2.

If v_1 and v_2 have a same color on all H_i, then a $(k-1)$-coloring of G can be obtained by combining $(k-1)$-colorings from all H_i. If v_1 and v_2 have a diffenrent color on all H_i, then a $(k-1)$-coloring of G can be obtained from all H_i. Moreover, if we can recoloring to obtain two above properties, then a $(k-1)$-coloring of G can be similarly obtained.

Supose that there are two subgraphs, say H_1, H_2, with the following properties.

- On H_1, vertices v_1 and v_2 have a same color and it is impossible to recoloring to obtain a $(k-1)$-coloring such that v_1 and v_2 have different colors.
- On H_2, vertices v_1 and v_2 have different colors and it is impossible to recoloring to obtain a $(k-1)$-coloring such that v_1 and v_2 have a same color.

Without loss of generality, suppose that a color on v_1, v_2 of H_1 is 1 and colors on v_1, v_2 of H_2 are 1 and 2. Since it is impossible to recoloring to obtain a $(k-1)$-coloring of H_1 such that v_1 and v_2 have different colors, there is a $(1, i)$-chain containing both v_1 and v_2 in H_1 for all $i = 2, 3, \ldots, k-1$. Hence, both v_1 and v_2 are adjecent to vertices with color i for all $i = 2, 3, \ldots, k-1$. That is, $d_{H_1}(v_1), d_{H_1}(v_2) \geq k-2$.

Since it is imposible to recoloring to obtain a $(k-1)$-coloring of H_2 such that both v_1 and v_2 has the same colors. Then there is a $(1,2)$-chain contaning both v_1 and v_2. Hence v_1 and v_2 are adjecent to vertices with color 2 and 1, respectively. For $i \geq 3$, vertices v_1 or v_2 must be adjacent to vertices with color i; otherwise, we can chance color on both v_1 and v_2 to i. That is, $d_{H_2}(v_1) + d_{H_2}(v_2) \geq k-1$.

Therefore $d(v_1) + d(v_2) = d_{H_1}(v_1) + d_{H_2}(v_1) + d_{H_1}(v_2) + d_{H_2}(v_2) \geq k-2+k-2+k-1 = 3k-5$.

□

Let G_2 and G_2 be two vertex disjoint graph, and let $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$ be two cliques with $|X_1| = |X_2| = k$. Let $f : X_1 \to X_2$ be a bijection, and let G by obtained from $G_1 \cup G_2$ by identifying x and $f(x)$ for every $x \in X_1$ and possibly deleting some edges with both ends in the clique of size k resulting from the identification. We say that G is a *k-sum* of G_1 and G_2.

Lemma 4.4. *If a graph G is a k-sum of G_1 and G_2, then $\chi(G) \leq \max\{\chi(G_1), \chi(G_2)\}$*

Theorem 4.2. *For $0 \leq t \leq 3$, the graphs with no K_{t+1} minor can be built by repeated clique-sum, starting from graphs with at most t vertices.*

Figure 4.2: Wagner graph V_8

Theorem 4.3 (Wagner). *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.*

Theorem 4.4 (Wagner). *A graph G does not contain K_5 as a minor if and only if G can be obtained from planar graph and V_8 by 0-sum, 1-sum, 2-sum and 3-sums.*

Theorem 4.5 (Wagner). *A graph contains no $K_{3,3}$ as a minor if and only if it can be obtained from planar graph and K_5 by 0-sum, 1-sum and 2-sums.*

4.2 Hadwiger's conejecture for $t \leq 5$

Conjecture 4 (Hadwiger's conjecture). For every integer $t \geq 0$, every graph with no K_{t+1}-minor is t-colorable.

Hadwiger's conjecture is proved for $t \leq 5$. In this section, we study how they proved this Hadwiger's conjecture.

When $t = 1$, a graph with no K_2-minor has no edge. Then it is clearly 1-colorable.

When $t = 2$, a graph with no K_3-minor is a tree. Then it is 2-colorable. The first nontrivial case for Hadwiger's conjecture is $t = 3$. Hadwiger proved that a non-null graph with no K_4-minor are 3-colorable. He proved that a graph with no K_4-minor has a vertex with degree at most 2 which implies that all such graphs are 3-colorable. Later, Dirac[12] and Duffin[14] are on the same topics.

Theorem 4.6. *A graph with no K_4-minor has a vertex with degree at most 2.*

Theorem 4.7. *Every 3-connected graph contains a K_4-minor.*

Proof. Let G be a 3-connected graph. Choose two distinct vertices $u, v \in V(G)$. Then there exsits 3 internal disjoint paths P, Q, R from u to v. At most one path can be an edge. Then we can find internal vertices $p \in V(P)$ and $q \in V(Q)$. Since G is 3-connected, there exsits a path between p, q in $G - u - v$, say S. Notice that S maybe intersects P, Q several times. Let S' be a shortest sub-path of S which is connected P and Q. If S' and R do not intersect, then $P \cup Q \cup R \cup S'$ is a K_4 subdivision in G. Otherwise, assume that S' and R share some vertices. Let S'' be the shorttest sub-path of S' which is connected P and R. then $P \cup Q \cup R \cup S''$ is a K_4 subdivision in G. □

The following is the shorter proof for Hadwiger's conjecture when $t = 3$.

Theorem 4.8. *[40] A graph with no K_4-minor is 3-colorable.*

Proof. Suppose there is a graph G that is not 3-colorable and has no K_4-minor. Select such a G with the minimum number of vertices. Clearly, G is connected and must contain a circuit. Pick an edge $e = v_1v_2$ on the circuit.

Let X be a minimum vertex set of $G - e$ seperating v_1 and v_2. Notice that X must be an independent set; otherwise, G would contain K_4 as a minor. Let $G - e$=$G_1 \cup G_2$ and $G_1 \cap G_2 = X$ such that G_1 and G_2 are connected and $v_1 \in G_1$ and $v_2 \in G_2$.

Let G_1' be the graph obtained from G by contracting all edges in G_1 so that all vertices of G_1 are identified with v_1. Notice that $V(G_1') = V(G_2 - X)) \cup \{v_1\}$ and $E(G_1') = E(G_2 - X) \cup \{v_1x$ when $x \in V(G_2 - X)$ is adjacent to a vertex of $G_1\}$. Define G_2', similarly.

Since G_1' and G_2' has no K_4-minor and both graphs are smaller than G. Both G_1 and G_2 are 3-colorable. Without loss of generality, suppose that v_1, v_2 from G_1 are colored by $1, 2$ and v_1, v_2 from G_2 are colored by $3, 1$.

Then 3-coloring of G can be obtained by the following steps.

- All veritces of $G_1 - X$ are colored similar to 3-coloring of G_2'.
- All vertices of $G_2 - X$ are colored similar to 3-coloring of G_1'.
- All vertices of X are colored by 1.

□

In 1937, Wagner[39] points out that four color problems implies Hadwiger's conjecture when $t = 4$. Theorem 4.9 is a proof of Haswiger's conjecture when $t = 4$.

Lemma 4.5. *Every non-planar graph 4-connected graph contains K_5 as a minor.*

Proof. Let G be a non-planar graph 4-connected graph. By Theorem 4.3 G contain K_5 or $K_{3,3}$ as a minor. If G does not contain $K_{3,3}$ as a minor, then G contains K_5 as a minor. Suppose that G contains $K_{3,3}$ as a minor. By Lemma 4.3, G contains a subdivision of $K_{3,3}$ as a subgraph, say H.

Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the vertices of H of degree three corresponding to vertices of two parts of bijection of $K_{3,3}$. For $x \in A$, let H_x denote the component of $H - B$ containing x, and for $y \in B$ let H_y denot the component of $H - A$ containing y.

Since G is 4-connected graph, $G - B$ is connected. we choose a path P in $G - B$ joining some two vertices in A such that $P \cup H$ is minimal. Without loss of generality, let the ends of P are vertices a, b.

Starting from vertex a, let x be the last vertex of P in H_a and let y be the first vertex of P in H_b. We devide P into three paths at x and y, say P_a, P', P_b. Clearly, P' is internally disjoint from H_a and H_b. By minimality of $P \cup H$, P_a and P_b are in H_a and H_b, respectively. We may assume that P' is disjoint from H because if P' intersect H_c, H_d, H_e of H_f, then we can find a shorter path which connects two vertices of A.

Since H_d, H_e, H_f are disjoint, P_a and P_b are in at most two of them. There exist a component from H_d, H_e, H_f which is internally disjoint from P. Without loss of generality, suppose that $H_d \cap P = \varnothing$.

Next, we choose a path Q in $G - A$ joining d to a vertex in B such that $Q \cup H$ is minimal. Let e be the other end of Q. Starting from vertex d, let x' and the last vertex of path Q in H_d and let y' be the first vertex of path Q in H_e. We devide Q into three paths at x' and y', say Q_d, Q', Q_e. Similar to P, Q_d and Q_e are in H_d and H_e and Q' is internally disjoint from H.

We claim that $H \cup P \cup Q$ contains a K_5-minor.

Case 1. P and Q are disjoint. Then we contract path P_a, P_b, Q_d and Q_e to obtain a subdivision of H_1 in Figure 4.3. Then we contract the edge cf to obtain K_5-minor.

Case 2. P and Q are not disjoint. Since $H_d \cap P = \varnothing$, we obtain that $P_a \cap Q_d = \varnothing$ and $P_b \cap Q_d = \varnothing$. Since Q_e intersects at most one of the paths P_a and P_b, suppose that $P_b \cap Q_e = \varnothing$. First, we contract all intersections to a single vertex and the we contract P_a, P_b, Q_d and Q_e except for the edges incident to a, b, d, e to obtain a subdivision of H_2 in Figure 4.3. Then we contract the edges af and ce to obtain K_5-minor. □

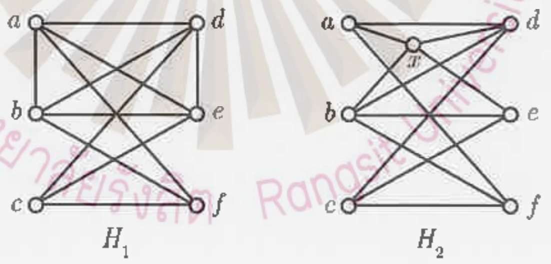


Figure 4.3: Extending a $K_{3,3}$ – *subdivision*

Theorem 4.9. *If a graph G does not contain K_5 as a minor, then G is 4-colorable.*

Proof. We will prove by induction on $|V(G)|$. Consider a nontrivial separation (A, B) of G of minimum order, and let $X = A \cap B$. By Theorem 4.5, G is not 4-connected. Then $|X| \leq 3$. We consider the case $|X = 3|$, the other case is easier.

Let G_1 and G_2 be graphs obtaining from $G[A]$ and $G[B]$, respectively by adding vertices z_1 and z_2, respectively adjacent to all vertices of X. Since (A, B) is minimum, G_1 and G_2 are minors of G. There fore, G_1 and G_2 do not contain K_5 as a minor.

Case 1. X is an independent set. Let G_i' for $i = 1, 2$ be the graph obtained from G_i by contracting edges $z_i x$ for all $x \in X$. By the induction hypothesis, both G_1' and G_2' are 4-colorable with all vertices of X receive the same color. Combining 4-colorings from G_1' and G_2' produce a 4-coloring of G.

Case 2. Some two vertices of X are adjacent. Let x be the remaining vertices. Let G_i'' for $i = 1, 2$ be the graph obtained from G_i by contracting edge $z_i x$. Notice that if we add som edges to make X a clique in $G[A]$ and $G[B]$, then we will obtain G_1'' and G_2'', respectively. By the induction hypothesis, both G_1'' and G_2'' are 4-colorable with all vertices of X receive distince colors. Combining 4-colorings from G_1' and G_2' produce a 4-coloring of G. □

A graph G is *apex* if $G - v$ is planar for some $v \in V(G)$. Robertson, seymour and Thomas established Hadwiger's conjecture for $t = 5$ by proving that minimum counter example is apex. The following conjecture would provide a more streamlined proof of their result.

Hadwiger's conjecture is open for $t \geq 6$.

4.3 Hadwiger's conjecture for $t \geq 6$

According to Paul Seymour in his recent survey on Hadwiger's conejecturee, the case $t \geq 6$ is still open. It is not know yet whether the graph with no K_7-minor is 6-colorable.

Kawarabayashi and Toft [20] proved that every graph contains neither K_7 nor $K_{4,4}$ as a minor is 6-colorable.

Jakobsen [17], [18] proved that every graph with no $K_7^{=}$-minor is 6-colorable and every graph with no K_7^{-}-minor is 7-colorable.

Albar and Gonçalves proved that every graph with no K_7-minor is 8-colorable and every graph with no K_8-minor is 10-colorable.

Rolek and Song [33] proved that every graph with no K_{t+1}-minor is $(2t-6)$-colorable for $t = 7, 8, 9$. They also proved every graph with no $K_8^{=}$-minor is 9-colorable and every graph with no K_8^{-}-minor is 8-colorable.

A graph G is *t-contraction-critical* if $\chi(G) = t$ and any proper minor of G is $(t-1)$-colorable.

Lemma 4.6. *[33] Every k-contraction cirtical graph G satisfies the following:*

1. *for any $v \in V(G), \alpha(G[N(v)]) \leq d(v) - k + 2$, where $\alpha(G[N(v)])$ denotes the independent number of the subgraph of G induced by $N(v)$;*

2. *no seperating set of G is a clique.*

Lemma 4.7. *[33] Let G be any k-contraction critical graph. Let $x \in V(G)$ be a vertex with degree $k+s$ with $\alpha(G[N(x)]) = s+2$ and let $S \subset N(x)$ with $|S| = s+2$ be any independent set, where $k \geq 4$ and $s \geq 0$ are integers.*

Let M be a set of missing edges of $G[N(X) - S]$. Then there exsits a collection $\{P_{uv} : uv \in M\}$ of paths in G such that for each $uv \in M$, P_{uv} has ends $\{u, v\}$ and all its intermal vertices in $G - N(x)$.

Moreover, if vertices u, v, w, z with $uv, wz \in M$ are distinct, then the paths P_{uv} and P_{wz} are vertex-disjoint.

Theorem 4.10. *[26] For $k \geq 7$, every k-contraction-critical graph is 7-connected.*

Lemma 4.8. *[33] For any 7-connected graph G, if G contains two different K_6-subgraph, then $G > K_8^-$.*

Theorem 4.11. *[25] For every integer $p = 1, 2, 3, \dots 7$, a graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

The edge bound in Theorem 4.11 is refered to as Mader's bound for the extremal function for K_p minors.

For graphs H_1, H_2 and an integer k, let us define an (H_1, H_2, k)-*cockade* recursively as follows. Any graph isomorphic to H_1 or H_2 is an (H_1, H_2, k)-cockade. Now suppose G_1 and G_2 be (H_1, H_2, k)-cockade. Let G be a graph obtained from disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2. Then the graph G is also an (H_1, H_2, k)-cockade.

If $H_1 = H_2 = H$, then G is simply called an (H, k)-cockade.

Theorem 4.12. *[19] Every graph on $n \geq 8$ vertices with at least $6n - 20$ edges either has a K_8 minor or is a $(K_{2,2,2,2,2}, 5)$-cockade.*

Theorem 4.13. *[36] Every graph on $n \geq 9$ vertices with at least $7n - 27$ edges has a K_8-minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.*

4.4 Hajós conjecture

Hajós Conjecture is a stronger conjecture than Hadwiger's conjecture. Hajós Conjecture says that every graph of chomatic number at least k contains a subdivision of K_k. Catlin [?] showed that Hajos conjecture fails for each $k \geq 7$, while $k \leq 4$, Dirac [12] verified the conjecture. Erdös and Fajtlowicz [15] showed that Hajos Conjecture failed for almost all graphs. Later, Thomassen [37] discovered many interesting counterexample to Hajos Conjecture. Mohar [28], Rodl and Zich [?] showed counterexamples from the view point of embeddings of graphs. For

$k = 5$, Yu and Zickfeld [41] proved that a possible minimal counterexample to the conjecture is 4-connected. Hajos conjecture is false for most graphs; however, it is true for some special classes of graphs. For example, the conjecture is true for the line graphs of simple graphs [38], graphs with large girth [21], [22]. In 2010, Deming Li, mingju Liu and Yumei Peng found a characterization for cycle power graphs C_n^k on Hajos Conjecture.

CHAPTER V

HADWIGER'S CONJECTRURE AND INFLATION OF 7-GRAPH

5.1 Main results

The main results of this reseach article is to prove that all inflations of 7-graph with minimum degree three satisfy Hadwiger's conjecture.

To prove the main result, we divide the prove into four parts; four sections. The first part is to stydy exist results about inflations of graphs. The results of the first part is all inflations of the following results satisfy Hadwiger's conjecture.

- a graph with $\chi(G) \leq 3$
- a graph with $\alpha(G) \leq 2$ and $n \leq 11$
- a perfect graph
- a graph builted from clique-sum without deleting edges from graphs satifying Hadwiger's conejecture.
- a join of two graphs satisfying Hadwiger's conjecture.

The second part is to prove that a 7-vertex graph with minimum degree three without the properties must be one of the four graph in Figure 5.1.

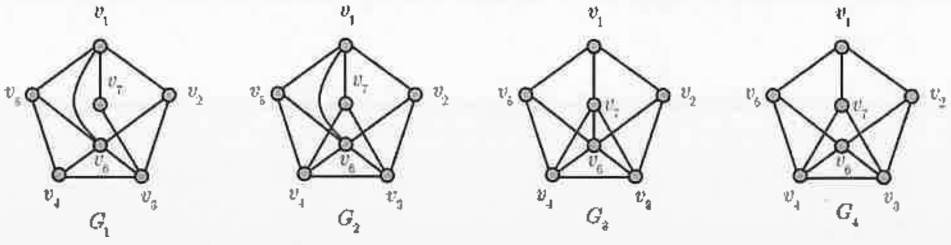


Figure 5.1: the four 7-graphs

The third part is to find the chomatic number of the four graphs. Let G_i be the graph with in the Figure 5.4 and the vertices of G_i be labelled as indicated for $i = 1, 2, 3, 4$. Let G_i' be an inflation of G_i where each vertex v_j has been inflated to a k_j-clique on a vertex set V_j for all j.

Theorem 5.1. $\chi(G_1') = \begin{cases} \max\{\omega(G_1'), \lceil \frac{n(G_1')+k_6-k_7}{2} \rceil\} & \text{if } k_7 \le k_2 + k_6 \\ \max\{\omega(G_1'), \lceil \frac{n(G_1')-k_2}{2} \rceil\} & \text{if } k_7 \ge k_2 + k_6 \end{cases}$

Theorem 5.2. $\chi(G_2') = \begin{cases} \max\{\omega(G_2'), \lceil \frac{n(G_2')+k_6-k_7}{2} \rceil\} & \text{if } k_7 \le \min\{k_2, k_5\} + k_6 \\ \max\{\omega(G_2'), \lceil \frac{n(G_2')-\min\{k_2,k_5\}}{2} \rceil\} & \text{if } k_7 \ge \min\{k_2, k_5\} + k_6 \end{cases}$

Theorem 5.3. $\chi(G_3') = \max\{\omega(G_3'), \lceil \frac{n(G_3')-k_7}{2} \rceil\}$.

Theorem 5.4. *Let* $M = k_1 + k_2 + k_5 + k_7$ *and* $N = k_3 + k_4 + k_6$.

Then $\chi(G_4') = \begin{cases} \max\{\omega(G_4'), \lceil \frac{n(G_4')-\min\{k_2,k_5,k_7\}}{2} \rceil\} & \text{if } \min\{k_2, k_5, k_7\} \le \frac{M-N}{3} \\ \max\{\omega(G_4'), \lceil \frac{n(G_4')+k_3+k_4+k_6}{3} \rceil\} & \text{if } \min\{k_2, k_5, k_7\} \ge \frac{M-N}{3} \end{cases}$

The fourth part is to prove that each of G_i' has a complete minor of order at least $\chi(G_i')$ for $i = 1, 2, 3, 4$. Hence, we canclude that all inflations of 7-vertex graph with minimum degree three satisfy Hadwiger's conjecture.

5.2 History

Given a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and positive integers $k_1, k_2, \ldots, k_n$, we define the *inflation* $G' = G(k_1, k_2, \ldots, k_n)$ of G to be graph obtained from G by replacing vertices $v_1, v_2, \ldots, v_n$ by disjoint cliques $A_1, A_2, \ldots, a_n$ of size $k_1, k_2, \ldots, k_n$, respectively, such that vertices of x and y where $x \in V(A_s)$ and $y \in V(A_t), s \neq t$ are adjacent if and only if v_s and v_t are adjacent in G.

The cliques $A_1, A_2, \ldots, A_n$ are referred to as the *inflated vertices*, and the numbers $k_1, k_2, \ldots, k_n$ as *inflation sizes* of G'. If $k_1 = k_2 = \ldots = k_n$, then G' is a *uniform* inflation. We also say that G' is obtained from *inflating* G.

A reason why we study Hadwiger's conjecture for inflations of graphs is from a counterexample of Hajós Conjecture which states that every k-chromatic graphs contains a subdivision of the complete graph with k vertices. In 1979 Calin [9] proved that Hajós Conjecture is false for $k \geq 6$. Surprisingly, Catlin's counterexaples are uniform inflations of the 5-cycle which are very simple.

According to the fact that if a graph G contains an H-minor, then G contains H-subdivition. Hence, if a graph satisfies Hajós conjecture, so is Hadwiger's conjecture. To find a counterexample to Hadwiger's conjecture, we will focus on a counterexample to Hajós conjecture.

There are some results on Hadwiger's conjecture for in flation of graphs. Pedersen [29] proved that there is no counterexample to Hadwiger's conjecture can be obtained from inflating the Petersen graph. Thomassen [37] proved that a graph G is perfect if and only if every inflation of G satisfied Hajós conjecture. That is, Hajós conjecture is true for every inflation of a perfect graph. Therefore, Hadwiger's conjecture is also true for every inflation of a perfect graph Plummer, Stiebitz and Toft stated that there is no counterexample to Hadwiger's conjecture can be obtained from inflating a graph with independence number at most 2 and order at most 11. Casselgren and Pedersen [8] prove that no counterexample to Hadwiger's conjecture can be obtained by inflating a 3-colorable graph. We can conclude that for a graph G with at most 11 vertices, If G is perfect or $\alpha(G) \leq 2$ or $\chi(G) \leq 3$, then G satisfies Hadwiger's conjecture.

5.3 Basic properties and examples

Recall that an inflation of C_5 is a counterexample to Hajós conjecture. Here, we will show that all inflation of C_5 satisfy Hawiger's conjecure. Let G' be an inflation of C_5. We will divide the proof into two parts. The first part is to show that $\chi(G') = \max\{\omega(G'), \lceil\frac{n(G')}{2}\rceil\}$. The second part is to show that G' has a complete minor of order $\max\{\omega(G'), \lceil\frac{n(G')}{2}\rceil\}$.

Lemma 5.1. *[29] Let G' be an inflation of C_5. Then $\chi(G') = \max\{\omega(G'), \lceil\frac{n(G')}{2}\rceil\}$.*

Proof. Suppose that each vertex v_i of $C_5 = v_1v_2v_3v_4v_5$ is inflated to a k_i- clique on a vertex set V_i of cardinality k_i, and let G' be a resulting graph. We will apply induction on $n(G')$. If $n(G') \leq 1$, then the statement holds. Suppose that $n(G') \geq 2$ and the statement holds for any inflation G'' of C_5 with $n(G'') < n(G')$.

Since G' is an inflation of C_5, there is no independent set of size 3. That is, at most two vertices have the same color. Then $\chi(G') \geq \lceil\frac{n(G')}{2}\rceil$. Obviously, $\chi(G') \geq \omega(G')$; hence, $\chi(G') \geq \max\{\omega(G'), \lceil\frac{n(G')}{2}\rceil\}$. It remains to show that $\chi(G') \leq \max\{\omega(G')$ or $\chi(G') \leq \lceil\frac{n(G')}{2}\rceil\}$.

Case 1. $\omega(G') \geq \lceil\frac{n(G')}{2}\rceil$. We will prove that G' has an $\omega(G')$-coloring to confirm that $\chi(G') \leq \omega(G')$. Without loss of generality, suppose that $k_1 + k_2 = \omega(G)$. Let A and B be sets of disjoint colors of size k_1 and k_2, respectively. According to the fact that $\omega(G) = k_1 + k_2$, we obtain $k_3 \leq k_1$ and $k_5 \leq k_2$. Let $A_1 \subseteq A$ and $B_1 \subseteq B$ of size k_3 and k_5, respectively. Since $\omega(G') \geq \lceil\frac{n(G')}{2}\rceil$, we obtain

$2k_1 + 2k_2 \geq k_1 + k_2 + k_3 + k_4 + k_5$. That is, $k_4 \leq (k_1 - k_3) + (k_2 - k_4)$.

Then there exsits an $\omega(G')$-coloring f of G' such that $f(V_1) = A, f(V_2) = B, f(V_3) = A_1, f(V_5) = B_1$ and $f(V_4) \subseteq (A \setminus A_1) \cup (B \setminus B_1)$. Hence, the statement holds.

Case 2. $\omega(G') \leq \lceil \frac{n(G')}{2} \rceil - 1$. Then G' is not a complete graph. Let u and v be nonadjacent vertices of G' and G'' be the graph obtained from deleting u, v from G'.

By induction hypothesis $\chi(G'') = \max\{\omega(G''), \lceil \frac{n(G'')}{2} \rceil\} = \lceil \frac{n(G'')}{2} \rceil$ because $\omega(G'') \leq \omega(G') \leq \lceil \frac{n(G')}{2} \rceil - 1 = \lceil \frac{n(G'')}{2} \rceil$.

A coloring of G' can be obtained from G'' by using a new color on u and v. Therefore, $\chi(G') \leq \chi(G'') + 1 = \lceil \frac{n(G'')}{2} \rceil + 1 = \lceil \frac{n(G')}{2} \rceil$. Hence, the statement holds. □

For the second part, we will show the stronger result.

Lemma 5.2. *[29] Let G' be an inflation of C_5 Then G' has a complete minor of order at least* $\frac{n(G')+\min\{k_1,k_2,k_3,k_4,k_5\}}{2}$.

Proof. Suppose that each vertex v_i of $C_5 = v_1v_2v_3v_4v_5$ is inflated to a k_i- clique on a vertex set V_i of cardinality k_i, and let G' be a resulting graph. Without loss of generality, let $k_1 = \min\{k_1, k_2, k_3, k_4, k_5\}$ and $k_2 + k_3 \geq k_4 + k_5$. Then $V_2 \cap V_3$ is a clique of size $k_2 + k_3$.

Let u be any vertex of V_1. Since $k_1 \leq k_2$ and $k_1 \leq k_3$, there are internal disjoint path from u to all veritces of V_3. According to definition of inflation, u is adjacent to all vertices of V_3. Hence G' has a complete minor of order $k_1 + k_2 + k_3 \geq \frac{n(G')+\min\{k_1,k_2,k_3,k_4,k_5\}}{2}$. □

A *clique-sum* is a way of combining two graphs by gluing them together at a clique. If two graphs G and H each contain cliques of equal size, the clique-sum of G and H is formed from their disjoint union by identifying pairs of vertices in these two cliques to form a single shared clique, and then possibly deleting some of the clique edges. A *k-clique-sum* is a clique-sum in which both cliques have at most k vertices If a graph G is obtained from clique-sum of a graph H_1 and H_2 without deleting edges, then $\chi(G) = \max\{\chi(H_1), \chi(H_2)\}$. That is, if both H_1 and H_2 satisfy Hadwiger's conjecture, so is G. Moreover, if all inflations of H_1 and H_2 satisfy Hadwiger's conjecture, so are inflations of G.

The *neighborhood* of v, denoted by $N(v)$ is the set of vertices adjacent to v. By applying clique-sum, if v is a vertex of a graph G such that $N(v)$ is a clique and $G - v$ satisfies Hadwiger's conjecture, then G also satisfies Hadwiger's conjecture.

The *join* of graphs G_1 and G_2, denoted by $G_1 \vee G_2$ is the graph with $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = \{uv : u \in V(G_1), v \in V(G_2)\}$. Observe that if G_1 has a-minor and G_2 has b-minor, the join graph $G_1 \vee G_2$ has a $(a+b)$-minor and $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$. Then if G_1 and G_2 satisfy Hadwiger's conjecture, so is $G_1 \vee G_2$. Moreover, if all inflations of G_1 and G_2 satisfy Hadwiger's conjecture, so are inflations of $G_1 \vee G_2$

A *wheel graph* is a graph formed by connecting a single universal vertex to all vertices of a cycle. In this paper, W_n denote a wheel graph with n vertices where $n \geq 4$. Since a wheel graph obtain from joining a cycle with a vertex, the wheel graph satisfies the Hawiger's conjecture.

Any inflation of a wheel graph can be builted from connected all vertices from a clique to all vertices of an inflation of a cycle. Since any inflation of a cycle satisfies Hadwiger's conjecture, so is any inflation of wheel graphs.

A *perfect graph* is a graph with the property that, in every one of its induced subgraphs, the size of the largest clique equals the minimum number of colors in a coloring of the subgraph. That is, a graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. The *strong perfect graph theorem* [10] states that A graph is perfect if and only if neither the graph nor its graph complement contains an odd graph cycle of length at least five as an induced subgraph. It was conjectured by Claude Berge in 1961. A proof by Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas was announced in 2002 and published by them in 2006

A perfect six-vertex graph with chromatic number at least four must be a wheel graph with six vertices. Then the graph has no independent of size 3. Hence, all six-vertex graphs satisfy Hadwiger's conjecture.

We can conclude that graphs satisfy Hadwiger's conjecture.

- a graph with $\chi(G) \leq 3$
- a graph with $\alpha(G) \leq 2$ and $n \leq 11$
- a perfect graph
- a graph builted from clique-sum without deleting edges from graphs satifying Hadwiger's conejecture.
- a join of two graphs satisfying Hadwiger's conjecture.

In this chapter, we will prove if G is a 7-vertex graph without the mension properties must be one of the five graphs in Figure 5.1. Then we find the chromatic number of the five graphs and prove that the five graphs satisfy Hadwiger's conjecture. Therefore, we can conclude that all inflations of a graph with at most seven vertices satisfies Hadwiger's conjecture.

5.4 The four 7-graph

Lemma 5.3. *Let G be a graph with at most seven vertices with the following properties.*

- $\chi(G) \geq 4$
- $\alpha(G) \geq 3$
- *imperfect*
- *there is no vertex v such that $N(v)$ is a clique*
- *there is no vertex with degree one and six.*

Then G must be a graph from Figure 5.1

Proof. Let G be a graph with at most seven vertices with all mention properties By the strong perfect graph theorem, the graph G or its complement $\overline{G}$ contains an odd cycle as an induced subgraph. Notice that G is not C_7 because of $\chi(G) \geq 4$ and $\overline{G}$ is not C_7 because of $\alpha(G) \geq 3$. Moreover, if G contain C_5 as an induced subgraph, so is $\overline{G}$. Therefore, the graph G contains C_5 as an induced subgraph. Let $C_5 = v_1v_2v_3v_4v_5$ and v_6, v_7 be the two remaining vertices of G such that $d(v_6) \geq d(v_7)$.

Observe that $\chi(G - S) \geq 3$ for all independent set $S \subset V(G)$. If there is an independent set S of size at least three containing both v_6 and v_7, the graph G is 3-colorable because $G - S$ is a subgraph of a path; contradiction. That is, an independent of size at least three must contain a vertex from v_6, v_7 and two vertices from v_1, v_2, v_3, v_4, v_5. Without loss of generality, assume that $S = \{v_2, v_5, v_7\}$ is an independent set of size 3. Hence, v_7 is adjacent to at most three vertices of C_5. Since G is not 3-colorable, $G - S$ is not a path for any independent set S. Then

$G-S$ must contain an odd cycle; the only possibility is $v_3v_4v_6$ is a triangle. That is, v_6 is adjacent to both v_4 and v_6

Case 1. v_6 is adjacent to all vertices of C_5. Then v_6 and v_7 are not adjacent because there is no vertex with degree six. If $d(v_7)=2$, then G is the first graph in Figure 5.1 because $N(v_7)$ is not a clique. If $d(v_7)=3$, then G is the second graph in Figure 5.1.

Case 2. v_6 is adjacent to all vertices of C_5 except v_1. If v_7 is not adjacent to v_1, then $d(v_7)=2$ and $G-v_7$ is not 3-colorable. That is, v_7 is adjacent to v_3, v_4, v_6. Therefire, $N(v_7)$ is a clique; contradiction. Hence, v_7 must be adjacent to v_1.

Since G is not 3-colorable, $G-S$ is not a path for any independent set S. Since $G-v_2-v_4-v_7$ is a path, we obtain that $\{v_2, v_4, v_7\}$ is not an indepensent set. That is, v_7 is adjacent to v_4. Similarly, we can conclude that v_7 is adjacent to v_3. If v_6 and v_7 are adjacent, then G is the third graph in Figure 5.1. If v_6 and v_7 are not adjacent, then G is the fourth graph in Figure 5.1.

□

5.5 The chromatic number of the four graphs

To find the chromatic number of an inflation of graphs from Figure 5.1, we need to find the chromatic number of an inflation of the four 6-vertex graph in Figure 5.2

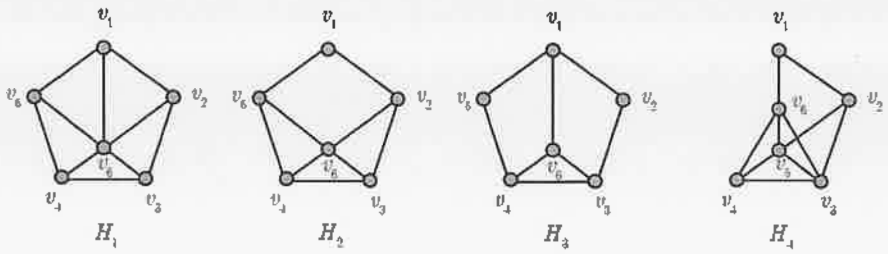


Figure 5.2: Four 6-vertex subraphs

For $j = 1, 2, 3, 4$, let H_j be the graph with in the Figure 5.2 and the vertices of GH be labelled as indicated and H'_j be an inflation of H_j where each vertex v_i has been inflated to a k_i-clique on a vertex set V_i for all i.

Lemma 5.4. $\chi(H'_1) = \max\{\omega(H'_1), \lceil \frac{n(H'_1)+k_6}{2} \rceil\}$.

Proof. Let H''_1 obtained from H'_1 by deleting V_6. Then $\chi(H'_1) = \chi(H''_1) + k_6$ and $\omega(H'_1) = \omega(H''_1) + k$ because each vertex of V_6 is adjacent to all remaining vertices. Since H''_1 is an inflation of C_5, we will apply Lemma 5.1,

$$\begin{aligned}\chi(H'_1) &= \chi(H''_1) + k \\ &= \max\{\omega(H''_1), \lceil \frac{n(H''_1)}{2} \rceil\} + k_6 \\ &= \max\{\omega(H''_1) + k_6, \lceil \frac{n(H''_1)}{2} \rceil + k_6\} \\ &= \max\{\omega(H'_1), \lceil \frac{n(H'_1) + k_6}{2} \rceil\}\end{aligned}$$

□

Lemma 5.5. $\chi(H'_2) = \max\{\omega(H'_2), \lceil \frac{n(H'_2)}{2} \rceil)\}$.

Proof. Since at most two vertices of H'_2 have the same color, we obtain $\chi(H'_2) \geq \lceil \frac{n(H'_2)}{2} \rceil$. Moreover, $\chi(H'_2) \geq \omega(H'_2)$ because a maximum clique of H'_2 requires $\omega(H'_2)$ colors. Then $\chi(H'_2) \geq \max\{\omega(H'_2), \lceil \frac{n(H'_2)}{2} \rceil\}$.

For every proper coloring f of H'_2, we have $f(V_6) \cap f(V_i) = \varnothing$ for $i = 2, 3, 4, 5$. To find the smallest number of colors, we will focus on a proper coloring f of H'_2 such that $f(V_1) \subseteq f(V_6)$ or $f(V_6) \subseteq f(V_1)$. Let $k = \min\{k_1, k_6\}$. We first use k colors to color vertices of A_1 and A_6. Let H''_2 denoted an induced subgraph of H'_2 induced by the set of uncolored vertices. Notice that $\omega(H''_2) + k = \omega(H'_2)$ because every maximal clique of H_2 contains either v_1 or v_6.

If $k_6 = k_1$, then H''_2 is a path; hence, $\chi(H''_2) = \omega(H''_2) \leq \max\{\omega(H'_2), \lceil \frac{n(H'_2)}{2} \rceil\}$. If $k_6 > k_1$, then H''_2 is an inflation of the graph Figure 5.3; hence, $\chi(H''_2) = \omega(H''_2) \leq \max\{\omega(H'_2), \lceil \frac{n(H'_2)}{2} \rceil\}$.

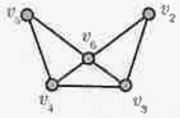


Figure 5.3: A figure for Lemma 5.5

Suppose that $k_6 < k_1$, then H'_2 is an inflation of C_5 ;hence, $\chi(H''_2) = \max\{\omega(H''_2), \lceil \frac{n(H''_2)}{2} \rceil\}$ by Lemma 5.1. Then $\chi(H'_2) \leq \max\{\omega(H''_2), \lceil \frac{n(H''_2)}{2} \rceil\} + k_6 = \max\{\omega(H'_2), \lceil \frac{n(H'_2)}{2} \rceil\}$. □

Lemma 5.6. *If $\omega(H'_2) = k_3 + k_4 + k_6 \leq k_1 + k_2 + k_5$, then $\chi(H'_2) = \lceil \frac{n(H'_2)}{2} \rceil$*

Proof. Assume that $\omega(H'_2) = k_3 + k_4 + k_6 \leq k_1 + k_2 + k_5$. Let k be an integer such that $k = \lceil \frac{k_1+k_2+k_5-k_3-k_4-k_6}{2} \rceil$. According to $\omega(H'_2) = k_3 + k_4 + k_6 \geq k_1 + k_2$, we obtain $k = \lceil \frac{k_1+k_2+k_5-k_3-k_4-k_6}{2} \rceil \leq= \lceil \frac{k_5}{2} \rceil < k_5$. Similarly, $k < k_2$.

We first color each k vertices of V_2 and V_5. Let H''_2 denote a subgraph of H''_3 induced by the set of uncolored vertices with new vertex sets $V'_2 \subseteq V_2$ and $V'_5 \subseteq V_5$ and cardinality k'_2 and k'_5. Then we obtain the following statements.

1. $k_1 \leq k_3 + k_4 + k_6 - k'_2 - k'_5$
2. $k'_2 < k_2 \leq k_4$
3. $k'_5 < k_5 \leq k_3$
4. $k + k_3 + k_4 + k_6 = \lceil \frac{n(H'_2)}{2} \rceil$

Let A, B and C be disjoint sets with cardinality k_3, k_4 and k_6. Let $A_1 \subseteq A$ and $B_1 \subseteq B$ be disjoint sets with cardinality, k'_5 and k'_2. Then there is a $(k_3 + k_4 + k_6)$-

coloring of H_2'' such that $f(V_3) = A, f(V_4) = B, f(V_6) = C$, $f(V_5') = A_1, f(V_2') = B_1$ and $f(V_1) \subseteq A \cup B \cup C \setminus A_1 \cup B_1$. Hence, $\chi(H_2') \leq k + k_3 + k_4 + k_6 = \lceil \frac{n(H_2')}{2} \rceil$.

Since at most two vertices of H_2' have the same color, we obtain $\chi(H_2') \geq \lceil \frac{n(H_2')}{2} \rceil$. Therefore, $\chi(H_2') = \lceil \frac{n(H_2')}{2} \rceil$. □

Lemma 5.7. $\chi(H_3') = \max\{\omega(H_3'), \lceil \frac{n(H_3') - \min\{k_2, k_5, k_6\}}{2} \rceil\}$.

Proof. Let $k = \min\{k_2, k_5, k_6\}$. Without loss of generality, suppose that $k_2 \geq k_5$. At most three vertices of H_3' have the same color and at most k colors can be used on three vertices. Then $\chi(H_3') \geq k + \lceil \frac{n(H_3') - 3k}{2} \rceil = \lceil \frac{n(H_3') - k}{2} \rceil$. Hence, $\chi(H_3') \geq \max\{\omega(H_3'), \lceil \frac{n(H_3') - k}{2} \rceil\}$.

We first color $k = \min\{k_2, k_5, k_6\}$ vertices from each of V_2, V_5, V_6. Let H_3'' denote a subgraph of H_3'' induced by uncolored vertices.

If $k = k_6$, then $\chi(H_3'') = \max\{\omega(H_3''), \lceil \frac{n(H_3'')}{2} \rceil\}$ by Lemma 5.1. Hence, $\chi(H_3') \leq \chi(H_3'') + k = \max\{\omega(H_3'') + k, \lceil \frac{n(H_3'')}{2} \rceil + k\} = \max\{\omega(H_3'), \lceil \frac{n(H_3') - k}{2} \rceil\}$.

Suppose that $k = k_5$. Then H_3'' is the graph wtih new vertex sets $V_2' \subseteq V_2, V_6' \subseteq V_6$ and cadinality k_2', k_6', respectively.

Case 1. $V_3 \cup V_4 \cup V_6'$ is a maximum clique. Then $k_1 \leq k_3 + k_4$, $k_4 + k_6' \geq k_2'$, $k_3 + k_4 + k_6' \geq k_1 + k_2'$ and $\omega(H_3'') \geq \lceil \frac{n(H_3'')}{2} \rceil$.

Let A, B and C denote disjoint sets of cadinality k_3, k_4 and k_6', respectively. Let A_1 be a set such that $A_1 \subseteq A$ if $k_1 \leq k_3$ and $A \subseteq A_1 \subseteq A \cup B$ if $k_1 > k_3$. Then there is an $\omega(H_3'')$-coloring of H_3'' such that $f(V_3) = A, f(V_4) = B, f(V_5') = C, f(V_1) = A_1$ and $f(V_2') \subseteq B \cup C \setminus A_1$.

Case 2. $V_1 \cup V_2'$ is a maximum clique. Then $k_3 \leq k_1$, $k_6' \leq k_2'$, $k_1 + k_2' \geq k_3 + k_4 + k_6'$ and $\omega(H_3'') \geq \lceil \frac{n(H_3'')}{2} \rceil$.

Let A and B denote disjoint sets of cadinality k_1 and k_2', respectively. Let $A_1 \subseteq A$ and $B_1 \subseteq B$ denote disjoint sets of cadinality k_3 and k_6'. respectively. Then there is an $\omega(H_3'')$-coloring of H_3'' such that $f(V_1) = A, f(V_2) = B, f(V_3) =$

$A_1, f(V_5') = B'$ and $f(V_4) \subseteq A \cup B \setminus A_1 \cup B_1$.

Case 3. $V_1 \cup V_6'$ is a maximum clique. Then $k_2' \leq k_6'$, $k_3 + k_4 \leq k_1$, $k_1 + k_6' \geq k_2' + k_3 + k_4$ and $\omega(H_3'') \geq \lceil \frac{n(H_3'')}{2} \rceil$.

Let A and B denote disjoint sets of cadinality k_1 and k_6', respectively. Then there is an $\omega(H_3'')$-coloring of H_3'' such that $f(V_1) = A, f(V_5') = B, f(V_2') \subseteq B$ and $f(V_3 \cup V_4) \subseteq A$.

Hence, we obtain that $\chi(H_3'') = \omega(H_3'') \geq \lceil \frac{n(H_3'')}{2} \rceil$. Notice that each maximal clique of H_3 contains exactly one vertex from v_2, v_5, v_6. Then $\omega(H_3') = \omega(H_3'') + k$.

Therefore, $\chi(H_3') \leq \chi(H_3'') + k = \omega(H_3'') + k = \omega(H_3') = \max\{\omega(H_3'), \lceil \frac{n(H_3')-k}{2} \rceil$.

□

Lemma 5.8. $\chi(H_4') = \omega(H_4') \geq \frac{n(H_4')}{2}$.

Proof. Notice that one clique from $V_1 \cup V_2$, $V_1 \cup V_6$, $V_2 \cup V_3 \cup V_5$ and $V_3 \cup V_4 \cup V_5 \cup V_6$.

Case 1. $V_1 \cup V_2$ is a maximum clique of H_4'. Then $k_2 \geq k_6$, $k_1 \geq k_3 + k_5$, $k_1 + k_2 \geq k_3 + k_4 + k_5 + k_6$ and $\omega(H_4') \geq \frac{n(H_4')}{2}$.

Let A and B denote disjoint sets of cadinality, k_1 and k_2. Let A_1 and A_2 denote disjoint fixed subset of A of cardinality, k_3 and k_5, respectively. Let B_1 denote fixed subset of B of cardinality, k_6. Then there is an $\omega(H_4')$-coloring f of H_4' with $f(V_1) = A$, $f(V_2) = B$, $f(V_3) = A_1$, $f(V_5) = A_2$, $f(V_6) = B_1$ and $f(V_4) \subseteq (A \cup B) \setminus (A_1 \cup A_2 \cup B_1)$.

Case 2. $V_1 \cup V_6$ is a maximum clique of H_4' Then $k_6 \geq k_2$, $k_1 \geq k_3 + k_4 + k_5$, $k_6 + k_1 \geq k_2 + k_3 + k_4 + k_5$ and $\omega(H_4') \geq \frac{n(H_4')}{2}$.

Let A and B denote disjoint sets of cadinality k_1 and k_6, respectively. Then there is an $\omega(H_4')$-coloring f of H_4' with $f(V_1) = A$, $f(V_6) = B$, $f(V_2) \subseteq B$ and $f(V_3 \cup V_4 \cup V_5) \subseteq A$.

Case 3. $V_2 \cup V_3 \cup V_5$ is a maximum clique of H_4'. Then $k_3 + k_5 \geq k_1$, $k_2 \geq k_4 + k_6$, $k_2 + k_3 + k_5 \geq k_1 + k_4 + k_6$ and $\omega(H_4') \geq \frac{n(H_4')}{2}$.

Let A, B and C denote disjoint sets of cadinality, k_2, k_3 and k_5, respectively. Then there is an $\omega(G')$-coloring f of H_4' with $f(V_2) = A$, $f(V_3) = B$, $f(V_5) = C$, $f(V_1) \subseteq B \cup C$ and $f(V_4 \cup V_6) \subseteq A$.

Case 4. $V_3 \cup V_4 \cup V_5 \cup V_6$ is a maximum clique of H_4'. Then $k_4 + k_6 \geq k_2$, $k_3 + k_4 + k_5 \geq k_1$, $k_1 + k_2 \geq k_3 + k_4 + k_5 + k_6$ and $\omega(H_4') \geq \frac{n(H_4')}{2}$

Let A, B and C denote disjoint sets of cadinality, $k_3 + k_5$, k_4 and k_6. Let C_1 be a set such that $C_1 \subseteq C$ if $k_2 \leq k_6$ and $C \subseteq B \cup C$ if $k_2 < k_6$. Then there is an $\omega(H_4')$-coloring f of H_4' with $f(V_3 \cup V_5) = A$, $f(V_4) = B$, $f(V_6) = C$, $f(V_2) = C_1$ and $f(V_1) \subseteq (A \cup B) \setminus C_1$. □

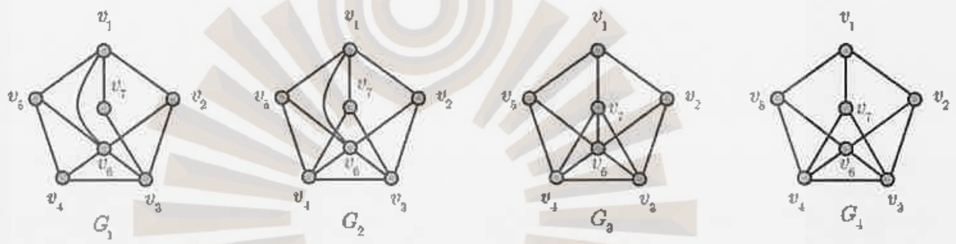


Figure 5.4: The four 7-vertex graphs

Let G_i be the graph with in the Figure 5.4 and the vertices of G_i be labelled as indicated for $i = 1, 2, 3, 4$. Let G_i' be an inflation of G_i where each vertex v_j has been inflated to a k_j-clique on a vertex set V_j for all j.

Theorem 5.1. $\chi(G_1') = \begin{cases} \max\{\omega(G_1'), \lceil \frac{n(G_1')+k_6-k_7}{2} \rceil\} & \text{if } k_7 \leq k_2 + k_6 \\ \max\{\omega(G_1'), \lceil \frac{n(G_1')-k_2}{2} \rceil\} & \text{if } k_7 \geq k_2 + k_6 \end{cases}$

Proof. Case 1. $k_7 \leq k_6 + k_2$. Then $V_1 \cup V_7$ and $V_3 \cup V_7$ are not a maximum clique. Hence, $\omega(G_1') = \omega(G_1' - V_7)$. If f a coloring of $G_1' - V_7$, then we can extend f to be coloring of G' by assigning $f(V_7) \subseteq f(V_2) \cup f(V_6)$. Hence, $\chi(G_1') = \chi(G_1' - V_7)$. By Lemma 5.4,

$$
\begin{aligned}
\chi(G_1') &= \chi(G_1' - V_7) \\
&= \max\{\omega(G_1' - V_7), \lceil \frac{n(G_1' - V_7) + k_6}{2} \rceil\} \\
&= \max\{\omega(G_1'), \lceil \frac{n(G_1') + k_6 - k_7}{2} \rceil\}
\end{aligned}
$$

Case 2. $k_7 > k_6 + k_2$. If f is a coloring of $G_1' - V_2$, then f can be extend to a coloring of G_1' by giving $f(V_2) = f(V_7) \setminus f(V_6)$. Then $\chi(G_1') = \chi(G_1' - V_2)$. By Lemma 5.5,

$$
\begin{aligned}
\chi(G_1') &= \chi(G_1' - V_2) \\
&= \max\{\omega(G_1' - V_2), \lceil \frac{n(G_1' - V_2)}{2} \rceil\} \\
&= \max\{\omega(G_1'), \lceil \frac{n(G_1') - k_2}{2} \rceil\}
\end{aligned}
$$

□

Corollary 5.9. $\chi(G_1') = \max\{\omega(G_1'), \lceil \frac{n(G_1')+k_6-k_7}{2} \rceil, \lceil \frac{n(G_1')-k_2}{2} \rceil\}$

Proof. It obtained from $k_2 + k_6 \geq k_7$ if and only if $\lceil \frac{n(G_1')+k_6-k_7}{2} \geq \rceil, \lceil \frac{n(G_1')-k_2}{2} \rceil$. □

Theorem 5.2. $\chi(G_2') = \begin{cases} \max\{\omega(G_2'), \lceil \frac{n(G_2')+k_6-k_7}{2} \rceil\} & \text{if } k_7 \leq \min\{k_2, k_5\} + k_6 \\ \max\{\omega(G_2'), \lceil \frac{n(G_2')-\min\{k_2,k_5\}}{2} \rceil\} & \text{if } k_7 \geq \min\{k_2, k_5\} + k_6 \end{cases}$

Proof. Notice that each vertex of V_6 is adjacent to all vertices except vertices of V_7. Let $k = \min\{k_6, k_7\}$. We color k vertices from each of V_6 and V_7 by k colors. Let G_2'' be the remainging graph. We obtain that $\omega(G_2') = \omega(G_2'') + k$ because every maximal clique of G_2 contains either v_6 or v_7

Case 1. $k = k_7 < k_6$. Then $k_7 \leq \min\{k_2, k_5\} + k_6$. By Lemma 5.4,

$$\begin{aligned}\chi(G') &= \chi(G_2'') + k_7 \\ &= \max\{\omega(G_2''), \lceil \frac{n(G_2'') + (k_6 - k_7)}{2} \rceil\} + k_7 \\ &= \max\{\omega(G_2'') + k_7, \lceil \frac{(n(G_2'') + 2k_7 + k_6 - k_7}{2} \rceil\} \\ &= \max\{\omega(G_2'), \lceil \frac{n(G_2') + k_6 - k_7}{2} \rceil\}\end{aligned}$$

Case 2. $k = k_6 \leq k_7$. By Lemma 5.7,

$$\begin{aligned}\chi(G') &\leq \chi(G_2'') + k_6 \\ &= \max\{\omega(G_2''), \lceil \frac{n(G_2'') - \min\{k_2, k_5, k_7 - k_6\}}{2} \rceil\} + k_6 \\ &= \max\{\omega(G_2'') + k_6, \lceil \frac{n(G_2'') + 2k_6) - \min\{k_2, k_5, k_7 - k_6\}}{2} \rceil\} \\ &= \max\{\omega(G_2'), \lceil \frac{n(G_2') - \min\{k_2, k_5, k_7 - k_6\}}{2} \rceil\}\end{aligned}$$

If $k_7 \leq \min\{k_2, k_5\} + k_6$, then $\chi(G_2'') = \max\{\omega(G_2'), \lceil \frac{n(G_2') + k_6 - k_7}{2} \rceil\}$. If $k_7 \geq \min\{k_2, k_5\} + k_6$, then $\chi(G_2'') = \max\{\omega(G_2'), \lceil \frac{n(G_2') - \min\{k_2, k_5\}}{2} \rceil\}$. □

Corollary 5.10. $\chi(G_2') = \max\{\omega(G_2'), \lceil \frac{n(G_2') + k_6 - k_7}{2} \rceil\}, \lceil \frac{n(G_2') - \min\{k_2, k_5\}}{2} \rceil\}$

Proof. It is obtained from $\lceil \frac{n(G_2') + k_6 - k_7}{2} \rceil\} \geq \lceil \frac{n(G_2') - \min\{k_2, k_5\}}{2} \rceil\}$ if and only if $k_7 \leq \min\{k_2, k_5\} + k_6$. □

Theorem 5.3. $\chi(G_3') = \max\{\omega(G_3'), \lceil \frac{n(G_3') - k_7}{2} \rceil\}$.

Proof. Without loss of generality, suppose that $k_2 \geq k_5$. Let $k = \min\{k_5, k_7\}$. We color k vertices from each of V_5 and V_7 by k colors. Let G_3'' be the remainging graph. We obtain that $\omega(G_3') = \omega(G_3'') + k$ because every maximal clique of G_3

contains either v_5 or v_7

Case 1. $k = k_5 < k_7$. By Lemma 5.8, $\chi(G_3') \leq \chi(G_3'') + k_5 = \omega(G_3'') + k_5 = \omega(G_3')$. Hence, $\chi(G_3') = \omega(G_3')$.

Moreover, $\omega(G_3'') + k_5 \geq \frac{n(G_3'')}{2} + k_5 = \frac{n(G_3')-3k_7}{2} + k_5 = \frac{n(G_3')-k_7}{2} + (k_5 - k_7) > \frac{n(G_3')-k_7}{2} \geq \lceil \frac{n(G_3')-k_7}{2} \rceil$. Therefore, $\chi(G_3') = \max\{\omega(G_3'), \lceil \frac{n(G_3')-k_7}{2} \rceil\}$.

Case 2. $k = k_7 \leq k_5$. By Lemma 5.5, $\chi(G_3') \leq \chi(G_3'')+k_7 = \max\{\omega(G_3''), \lceil \frac{n(G_3'')}{2} \rceil)\}+ k_7 = \max\{\omega(G_3'), \lceil \frac{n(G_3')-k_7}{2} \rceil\}$. Since at most three vertices have the same color and at most k_7 colors can appears on three vertices, we obtain $\chi(G_3') \geq k_7 + \frac{n(G_3')-3k_7}{2} = \frac{n(G_3')-k_7}{2}$. Then, $\chi(G_3') \geq \max\{\omega(G_3'), \lceil \frac{n(G_3')-k_7}{2} \rceil\}$. Therefore, $\chi(G_3') = \max\{\omega(G_3'), \lceil \frac{n(G_3')-k_7}{2} \rceil\}$. □

Theorem 5.4. *Let $M = k_1 + k_2 + k_5 + k_7$ and $N = k_3 + k_4 + k_6$.*

$$\text{Then } \chi(G_4') = \begin{cases} \max\{\omega(G_4'), \lceil \frac{n(G_4')-\min\{k_2,k_5,k_7\}}{2} \rceil\} & \text{if } \min\{k_2, k_5, k_7\} \leq \frac{M-N}{3} \\ \max\{\omega(G_4'), \lceil \frac{n(G_4')+k_3+k_4+k_6}{3} \rceil\} & \text{if } \min\{k_2, k_5, k_7\} \geq \frac{M-N}{3} \end{cases}$$

Proof. Without loss of generality, suppose that $k_7 = \min\{k_2, k_5, k_7\}$.

Case 1. $3k_7 \leq M - N$. Observe that at most three vertices of G_4' have the same color and at most k_7 colors can be used on three vertices. To color all vertices of G_4', we need at least $k_7 + \lceil \frac{n(G_4')-3k_7}{2} \rceil = \lceil \frac{n(G_4')-k_7}{2} \rceil$. Hence, $\chi(G_4') \geq \max\{\omega(G_4'), \lceil \frac{n(G_4')-k_7}{2} \rceil\}$.

We first use k_7 colors to color each vertices from each V_2, V_5 and V_7. Let G_4'' be a subgraph of G_4' induced by uncolored vertices with new vertex sets V_2', V_5' and cardinality, k_2', k_5'. Since $3k_7 \leq k_1 + k_2 + k_5 + k_7 - k_3 - k_4 - k_6$, we obain $k_1 + k_2' + k_5' \geq k_3 + k_4 + k_6$ and G_4'' is isomorphic to H_2'.

If $V_3 \cup V_4 \cup V_6$ is a maximum clique of G_4'', then we apply Lemma 5.6 to obtain $\chi(G_4') \leq \chi(G_4'')+k_7 \leq \lceil \frac{n(G_4'')}{2} \rceil+k_7 = \lceil \frac{n(G_4')-k_7}{2} \rceil \leq \lceil \frac{n(G_4')-k_7}{2} \rceil \leq \lceil \max\{\omega(G_4'), \lceil \frac{n(G_4')-k_7}{2} \}\rceil$.

Otherwise, $\omega(G_4') = \omega(G_4'') + k_7$ and we apply Lemma 5.5 to obtain $\chi(G_4') \leq \chi(G_4'') + k_7 = \max\{\omega(G_4'') + k_7, \lceil \frac{n(G_4'')}{2} \rceil + k_7\} = \lceil \max\{\omega(G_4'), \lceil \frac{n(G_4')-k_7}{2} \}\rceil$.

Case 2. $3k_7 > M - N$. At most three vertices of G_4' have the same color and at most k_7 colors can be used on three vertices and $V_2 \cup V_4 \cup V_6$ requires $k_2 + k_4 + k_6$ colors.

$$\begin{aligned}
\chi(G_4') &\geq k_7 + k_2 + k_4 + k_6 \\
&= \lceil \frac{(k_3 + k_4 + k_6 + 3k_7) + (2k_2 + 2k_4 + 2k_6)}{3} \rceil \\
&> \lceil \frac{(k_1 + k_2 + k_5) + (k_7 + 2k_3 + 2k_4 + 2k_6)}{3} \rceil \\
&= \lceil \frac{(n(G_4')) + k_3 + k_4 + k_6}{3} \rceil
\end{aligned}$$

Hence, $\chi(G_4') \geq \max\{\omega(G_4'), \lceil \frac{n(G_4') + k_3 + k_4 + k_6}{3} \rceil\}$

Let $k = \lceil \frac{n(G') - 2k_3 - 2k_4 - 2k_6}{3} \rceil$. Then $k = \lceil \frac{k_1 + k_2 + k_5 + k_7 - k_3 - k_4 - k_6}{3} \rceil = \lceil \frac{M-N}{3} \rceil < k_7$. First, we use k color to color each k vertices from each V_2, V_5 and V_7. Let G_4'' be the remaining graph with new vertex set V_2', V_5' and V_7' with cardinality k_2', k_5' and k_7', respectively. Since $3k_7 > M - N$, we obtain $k_1 + k_2' + k_5' + k_7' \leq k_2 + k_4 + k_6$.

Without loss of generality, suppose that $V_2' \cup V_4 \cup V_6$ or $V_3 \cup V_4 \cup V_6$ is a maximum clique of G_4''.

Case 2.1. $V_2' \cup V_3 \cup V_6$ is a maximum clique of G_4''. Moreover, we obtain $V_2 \cup V_4 \cup V_6$ is a maximum clique of G_4' and $\omega(G_4') = \omega(G_4'') + k$. Let A, B and C be disjoint sets of cadinality k_2', k_3 and k_6. Let $A' \subseteq A$ of cardinality k_4.

Observe that $k_5' \leq k_3$ or $k_7' \leq k_6$ because $k_1 + k_2' + k_5' + k_7' \leq k_2 + k_4 + k_6$. Without loss of generality, suppose that $k_5' \leq k_3$, let $B' \subseteq B$ of cadinality k_5'. Then there is an $\omega(G_4'')$-coloring of G_4'' with $f(V_2') = A$, $f(V_3) = B$, $f(V_6') = C$, $f(V_4) = A'$, $f(V_5') = B'$ and $f(V_1 \cup V_7) = A' \cup B \cup C \setminus A \cup B'$ Hence, $\chi(G_4') \leq \chi(G_4'') + k \leq \omega(G_4'') + k = \omega(G_4')$ Therefore, $\chi(G_4') = \omega(G'') \leq \max\{\omega(G_4'), \lceil \frac{n(G_4') + k_3 + k_4 + k_6}{3} \rceil$.

Case 2.2. $V_3 \cup V_4 \cup V_6$ is a maximum clique of G_4'' Then $k_2' \leq k_4, k_5' \leq k_3$ and

$k_7' \leq k_6$. Let A, B and C be disjoint sets of cadinality k_3, k_4 and k_6. Let $A' \subseteq A, B' \subseteq B$ and $C' \subseteq C$ of cardinality k_2', k_5' and k_7' Then there is an $(k_3+k_4+k_6)$-coloring of G_4'' with $f(V_3) = A, f(V_4) = B, f(V_6) = C$, $f(V_2') = A'$, $f(V_5') = B'$, $f(V_7') = C'$ and $f(V_1) = A \cup B \cup C \setminus A' \cup B' \cup C'$. Since $k_3 + k_4 + k_6$ is the size of a clique of G_4'', we obtain that $\chi(G_4'') = k_3 + k_4 + k_6$

$$\begin{aligned}\chi(G_4') &\leq \chi(G_4'') + k \\ &= k_2 + k_4 + k_6 + \lceil \frac{n(G_4') - 2k_3 - 2k_4 - 2k_6}{3} \rceil \\ &= \lceil \frac{n(G_4') + k_3 + k_4 + k_6}{3} \rceil \\ &\leq \max\{\omega(G_4'), \lceil \frac{n(G_4') + k_3 + k_4 + k_6}{3} \rceil\end{aligned}$$

□

Corollary 5.11. $\chi(G_4') = \max\{\omega(G_4'), \lceil \frac{n(G_4') - \min\{k_2,k_5,k_7\}}{2} \rceil, \lceil \frac{n(G_4') + k_3 + k_4 + k_6}{3} \rceil\}$.

Proof. It is obtained from $\lceil \frac{n(G_4') - \min\{k_2,k_5,k_7\}}{2} \rceil \geq \lceil \frac{n(G_4') + k_3 + k_4 + k_6}{3} \rceil$ if and only if $k_3 + k_4 + k_6 + 2k_7 \leq k_1 + k_2 + k_5$. □

Corollary 5.12. *If $\omega(G_4') = k_2 + k_3 + k_6$, then $\chi(G_4') \leq k_1 + k_2 + k_3 + k_6$.*

Proof. Let A, B, C and D be disjoint sets of cadinality k_2, k_3, k_6 and k_1. Let $A' \subseteq A$ be a set of cardinality k_4 and let $B' \subseteq A \cup B \setminus A'$ be a set of cardinality B' Let $C' \subseteq A \cup C \setminus A'$ be a set of cardinality C'. Then there is an $(k_1 + k_2 + k_3 + k_6)$-coloring of G_4' with $f(V_2) = A, f(V_3) = B, f(V_6) = C, f(V_1) = D$, $f(V_4) = A', f(V_5) = B'$ and $f(V_7') = C'$. □

5.6 Complete minor

Lemma 5.13. *Let H_1 be the graph with in the Figure 5.2 and the vertices of H_1 be labelled as indicated. Then H_1' has a complete minor of order at least $\frac{n(H_1') + k_6 + \min\{k_1,k_2,k_3,k_4,k_5\}}{2}$*

Proof. By Lemma 5.2, $H_1'-V_6$ has a complete minor of order $\frac{n(H_1'-V_6)+\min\{k_1,k_2,k_3,k_4,k_5\}}{2}$. Since each vertex of V_6 is adjacent to all remaining vertices, the graph H_1' has a minor of order $\frac{n(H_1'-V_6)+\min\{k_1,k_2,k_3,k_4,k_5\}}{2}+k_6=\frac{n(H_1')+k_6+\min\{k_1,k_2,k_3,k_4,k_5\}}{2}$. □

Theorem 5.5. *Hadwiger's conjecture holds for any inflation of C_5.*

Proof. Let G be an inflation of C_5 By Lemma 5.1, we obtain that $\chi'(G') \leq \max\{\omega(G'), \lceil\frac{n(G')}{2}\rceil\}$. Obviously, G' has a complete graph of order $\omega(G')$ as a subgraph. Then G' has a complete minor of order $\omega(G')$. By Lemma 5.2, G' has a complete minor of order $\frac{n(G')+\min\{k_1,k_2,k_3,k_4,k_5\}}{2} \geq \lceil\frac{n(G')}{2}\rceil$. Hence, G' has a complete minor of order at least $\chi(G')$. □

Lemma 5.14. *Then H_2' has a complete minor of order at least $\lceil\frac{n(H_2')}{2}\rceil$.*

Proof. Without loss of generality, suppose one clique of $V_1\cup V_2, V_2\cup V_3\cup V_6, V_3\cup V_4\cup V_6$ is a maximum clique.

Case 1. $V_1\cup V_2$ is a maximum clique. Then we obtain the following statements.

1. $k_1+k_2\geq k_3+k_4+k_6$; hence, $k_1+k_2+k_5\geq k_3+k_4+k_6$
2. $k_1\geq k_3+k_6$ and $k_2\geq k_5$; hence, $k_1+k_2+k_4\geq k_3+k_5+k_6$
3. $k_1+k_2\geq k_4+k_5+k_6$; hence, $k_1+k_2+k_3\geq k_4+k_5+k_6$

Let V_a is a smallest set from V_3,V_4,V_5. Then there are internal disjoint path from all vertices of V_a to all vertices of $V_1\cup V_2$. That is, there is a complete minor of order $k_1+k_2+k_a$. From the three above statements $k_1+k_2+k_a\geq\lceil\frac{n(H_2')}{2}\rceil$.

Case 2. $V_2\cup V_3\cup V_6$ is a maximum clique. Then we obtain the following statements.

1. $k_2+k_3+k_6\geq k_1+k_5$; hence, $k_2+k_3+k_4+k_6\geq k_1+k_5$
2. $k_2+k_3+k_6\geq k_4+k_5$; $k_1+k_2+k_3+k_6\geq k_4+k_5$

3. $k_2 \geq k_4$ and $k_3 + k_6 \geq k_1$; hence, $k_2 + k_3 + k_5 + k_6 \geq k_1 + k_4$

Similar to Case 1, let V_b is a smallest set from V_1, V_4, V_5. Then there are internal disjoint path from all vertices of V_a to all vertices of $V_2 \cup V_3 \cup V_6$. That is, there is a complete minor of order $k_2 + k_3 + k_5 + k_b$. From the three above statements $k_2 + k_3 + k_6 + k_b \geq \lceil \frac{n(H_2')}{2} \rceil$.

Case 3. $V_3 \cup V_4 \cup V_6$ is a maximum clique. Then we obtain the following statements.

1. $k_3 + k_4 + k_6 \geq k_1 + k_2$; hence, $k_3 + k_4 + k_5 + k_6 \geq k_1 + k_2$

2. $k_3 + k_4 + k_6 \geq k_1 + k_5$; hence, $k_2 + k_3 + k_4 + k_6 \geq k_1 + k_5$

3. $k_3 \geq k_5$ and $k_4 \geq k_2$; hence, $k_3 + k_4 + k_5 + k_6 \geq k_1 + k_2$

Similar to Case 1 and 2, let V_c is a smallest set from V_1, V_2, V_5. Then there are internal disjoint path from all vertices of V_c to all vertices of $V_3 \cup V_4 \cup V_6$. That is, there is a complete minor of order $k_3 + k_4 + k_6 + k_c$. From the three above statements $k_3 + k_4 + k_6 + k_c \geq \lceil \frac{n(H_2')}{2} \rceil$ □

Theorem 5.6. *The four graphs from Figure 5.1 satisfy Hadwiger's conjecture.*

Proof. Let G be a graph in the Figure 5.1 and the vertices of G be labelled as indicated. Let G' be an inflation of G where each vertex v_i has been inflated to a k_i-clique on a vertex set V_i for all i.

If $\chi(G') = \omega(G')$, then it is obvious that G_1' has a minor of order $\chi(G')$; hence, G' satisfies Hadwiger's conjecture. Suppose that $\chi(G') > \omega(G')$.

Case 1. G is G_1 or G_2. If G is G_2, then we assume that $k_2 \leq k_5$. Then $G - V_7$ is H_1. By Lemma 5.13, H_1' has a complete minor of order $\lceil \frac{n(H_1')+k_6}{2} \rceil$.

$$\lceil\frac{n(H_1')+k_6}{2}\rceil=\lceil\frac{n(G'-V_7)+k_6}{2}\rceil$$
$$=\lceil\frac{n(G')+k_6-k_7}{2}\rceil$$

If $k_7 \geq k_2+k_6$, then H_1' has a complete minor of order $\lceil\frac{n(G'-V_7)+k_6}{2}\rceil = \lceil\frac{n(G')+k_6-k_7}{2}\rceil \geq \lceil\frac{n(G')-k_2}{2}\rceil$. Hence, $G'-V_7$ has a complete minor of order $\chi(G')$. That is, G' has a complete minor of order $\chi(G')$.

Case 2. G is G_3 or G_4. If G is G_4, then we assume that $k_7 \leq k_2, k_5$. Then $G-V_7$ is H_2. By Lemma 5.14, H_2' has a complete minor of order $\lceil\frac{n(H_2')}{2}\rceil = \lceil\frac{n(G)-k_7}{2}\rceil$. By Theorem 5.3 and by Theorem 5.4, $\chi(G') = \frac{n(G)-k_7}{2}$ unless G' is G_4' and $k_3+k_4+k_6+2k_7 \geq k_1+k_2+k_5$.

Suppose that $G'=G_4'$ and $k_3+k_4+k_6+2k_7 \geq k_1+k_2+k_5$. By Theorem 5.4,

$$\chi(G_4') \leq \lceil\frac{n(G')+k_3+k_4+k_6}{3}\rceil$$
$$\leq \lceil\frac{k_1+k_2+k_5+k_7+2k_3+2k_4+2k_6}{3}\rceil$$
$$\leq \lceil\frac{k_3+k_4+k_6+2k_7+k_7+2k_3+2k_4+2k_6}{3}\rceil$$
$$= k_3+k_4+k_6+k_7$$

Case 2.1 $k_1 \geq k_7$. There are k_7 internal disjoint path from all vertices of V_1 to all vertices of $V_3 \cup V_4 \cup V_6$. Hence, there is a complete minor of order $k_3+k_4+k_6+k_7$.

Case 2.2 $k_1 \leq k_7$ and $k_1 \geq k_4$. There are internal disjoint path from all vertices of V_4 to all vertices of $V_2 \cup V_3 \cup V_6$. That is, there is a complete minor of order $k_2+k_3+k_4+k_6 \geq k_3+k_4+k_6+k_7$ because of $k_2 \geq k_7$.

Case 2.2 $k_1 \leq k_7$ and $k_1 \leq k_4$. If $V_2 \cup V_3 \cup V_6$ is a maximum clique of G_4', then

$\chi(G'_4) \leq k_1 + k_2 + k_3 + k_6$ by Corollary 5.12. If $V_3 \cup V_4 \cup V_6$ is a maximum clique of G'_4, then $\chi(G'_4) \leq k_1 + k_3 + k_4 + k_6$. Since $k_1 \leq k_2, k_4, k_5, k_7$. There are internal disjoint path from all vertices of V_1 to all vertices of $V_2 \cup V_3 \cup V_6$. Therefore, there is a complete minor of order $k_1 + k_2 + k_3 + k_6$. Similarly, there is a complete minor of order $k_1 + k_3 + k_4 + k_6$. □

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